Mathematical and numerical analysis for two phase flows in porous media

Ziad Khalil, Mazen Saad

Ecole Centrale de Nantes
Laboratoire de Mathématiques Jean Leray

Plan

Mathematical Analysis for two compressible phase flow

Convergence of finite volume scheme for water–gas flow

Model

The equations describing the immiscible diplacement of two compressible phases are

Mass conservation of each phase:

$$\phi \partial_t (\rho_i(p_i) s_i) + \mathrm{div}(\rho_i(p_i) \mathbf{V}_i) + \rho_i s_i f_P = \rho_i s_i^I f_I, \qquad i = 1, 2$$

Saturation : $s_i(t,x)$
Pressure: $p_i(t,x)$
Porosity: $\phi(x)$

Density: $\rho_i(p_i)$
Injection term: $f_I(t,x)$
Production term: $f_P(t,x)$

Known saturation: $s_i^I(t,x)$

Model

The equations describing the immiscible diplacement of two compressible phases are
Mass conservation of each phase:

$$\phi \partial_t(\rho_i(p_i)s_i) + \mathrm{div}(\rho_i(p_i)\mathbf{V}_i) + \rho_i s_i f_P = \rho_i s_i^I f_I, \qquad i = 1, 2$$

Definition of saturations

$$s_1 + s_2 = 1.$$

Darcy law for velocity $\mathbf{V}_i$

$$\mathbf{V}_i = -\mathbf{K} M_i(s_i)(\nabla p_i - \rho_i(p_i)\mathbf{g}), \qquad i = 1, 2,$$

Permeability tensor: $\mathbf{K}$

Mobility: $M_i(s_i) = k_i(s_i)/\mu_i$

Viscosity: μ_i

Gravity: $\mathbf{g}$

Model

The equations describing the immiscible diplacement of two compressible phases are

Mass conservation of each phase:

$$\phi\partial_t(\rho_i(p_i)s_i) + \mathrm{div}(\rho_i(p_i)\mathbf{V}_i) + \rho_i s_i f_P = \rho_i s_i^I f_I, \qquad i = 1, 2$$

Definition of saturations

$$s_1 + s_2 = 1.$$

Darcy law for velocity $\mathbf{V}_i$

$$\mathbf{V}_i = -\mathbf{K} M_i(s_i)(\nabla p_i - \rho_i(p_i)\mathbf{g}), \qquad i = 1, 2,$$

Density

$\rho_i = \rho_i(p_i)$ increasing and bounded function ($0 < \rho_m \le \rho_i(p_i) \le \rho_M$).

Capillary pressure law

$f(s_1) = p_1 - p_2$ increasing function.

Problem

We search for (p_1, p_2) solution of the system

$$\phi\partial_t(\rho_i(p_i)s_i) - \mathrm{div}(\mathbf{K}\rho_i(p_i)M_i(s_i)\nabla p_i) + \mathrm{div}(\mathbf{K}\rho_i^2(p_i)M_i(s_i)\mathbf{g}) \\ + \rho_i(p_i)s_i f_P = \rho_i(p_i)s_1^I f_I, \quad i = 1, 2$$

$$s_1 + s_2 = 1$$

$$f(s_1) = p_1 - p_2$$

- Boundary conditions

$$p_1(t,x) = 0, \quad p_2(t,x) = 0 \text{ on } \Gamma_1 \text{ (injection boundary)}$$

$$\mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 \text{ on } \Gamma_{imp} \text{ (imprevious boundary)}$$

- Initial conditions

$$p_1(0,x) = p_1^0(x), \quad p_2(0,x) = p_2^0(x) \quad \text{in } \Omega$$

Degeneracy

degeneracy in the evolution term: $s_i = 0$
degeneracy in the diffusion term: $M_i(0) = 0$

$$\phi\partial_t(\rho_i(p_i)s_i) - \mathrm{div}(\mathbf{K}\rho_i(p_i)M_i(s_i)\nabla p_i)$$

$$+\,\mathrm{div}(\mathbf{K}\rho_i^2(p_i)M_i(s_i)\mathbf{g}) + \rho_i(p_i)s_i f_P(t,x) = \rho_i(p_i)s_1^I f_I(t,x), \quad i = 1,2. \tag{1}$$

- overcoming diffusion term degeneracy by adding η term,

$$\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta) - \mathrm{div}(\mathbf{K}\rho_i(p_i^\eta)M_i(s_i^\eta)\nabla p_i^\eta) + \eta(-1)^i\,\mathrm{div}(\rho_i(p_i^\eta)\nabla f(s_1^\eta))$$

$$+\,\mathrm{div}(\mathbf{K}\rho_i^2(p_i^\eta)M_i(s_i^\eta)\mathbf{g}) + \rho_i(p_i^\eta)s_i^\eta f_P = \rho_i(p_i^\eta)s_1^I f_I, \quad i = 1,2. \tag{2}$$

- overcoming evolution term degeneracy by a time discretization method,

$$\frac{\rho_i(p_i)s_i - \rho_i^\star s_i^\star}{h} - \mathrm{div}(\mathbf{K}\rho_i(p_i)M_i(s_i)\nabla p_i) + \ldots \tag{3}$$

Main estimate : Global pressure

The global pressure p is defined as

$$p = p_1 + \bar{p}(s_1) = p_2 + \tilde{p}(s_1),$$

where

$$\frac{d\bar{p}}{ds}(s_1) = -\frac{M_2(s_2)}{M(s_1)}\frac{df}{ds}(s_1) \quad \text{and} \quad \frac{d\tilde{p}}{ds}(s_1) = \frac{M_1(s_1)}{M(s_1)}\frac{df}{ds}(s_1).$$

Consequently, we have

$$\nabla p_1 = \nabla p + \frac{M_2}{M}\nabla f(s_1) \text{ and } \nabla p_2 = \nabla p - \frac{M_1}{M}\nabla f(s_1),$$

and

$$M(s_1)|\nabla p|^2 + \frac{M_1 M_2}{M}|\nabla f(s_1)|^2 = M_1(s_1)\nabla p_1 \cdot \nabla p_1 + M_2(s_2)\nabla p_2 \cdot \nabla p_2.$$

Define, the function β such that $\frac{M_1^2 M_2^2}{M^2}|\nabla f(s_1)|^2 = |\nabla \beta(s_1)|^2$.

The key point is to obtain the estimates on ∇p *and* $\nabla \beta(s_1)$.

Main estimate : evolutive terms

Define the test function

$$g_i(p_i) := \int_0^{p_i} \frac{1}{\rho_i(\xi)}\, d\xi.$$

And

$$\mathcal{H}_i(p_i) := \rho_i(p_i) g_i(p_i) - p_i,$$

Note that

$$\mathcal{H}_i(0) = 0,\ \mathcal{H}_i(p_i) \geq 0,\ |\mathcal{H}_i(p_i)| \leq c|p_i| \quad \text{for all } p_i.$$

We have the equality :

$$\partial_t(\rho_1(p_1)s_1)g_1(p_1) + \partial_t(\rho_2(p_2)s_2)g_2(p_2) = \partial_t\Big(s_1\mathcal{H}_1(p_1) + s_2\mathcal{H}_2(p_2)\Big) + \partial_t\Big(\int_0^{s_1} f(\xi)\, d\xi\Big).$$

Note that

$$s_1\mathcal{H}_1(p_1) + s_2\mathcal{H}_2(p_2) \geq 0 \text{ and } \int_0^{s_1} f(\xi) \text{ is bounded.}$$

Main estimate

$$\phi\partial_t(\rho_i(p_i)s_i) - \operatorname{div}(\mathbf{K}\rho_i(p_i)M_i(s_i)\nabla p_i) +$$

$$+ \operatorname{div}(\mathbf{K}\rho_i^2(p_i)M_i(s_i)\mathbf{g}) + \rho_i(p_i)s_i f_P = \rho_i(p_i)s_1^I f_I$$

Multiplying by $g_i(p_i) = \int_0^{p_i} \frac{1}{\rho_i(\xi)}\, d\xi$, integrating over Ω and adding terms for $i = 1, 2$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi\,(s_1\mathcal{H}_1(p_1) + s_2\mathcal{H}_2(p_2))\,dx + \frac{d}{dt}\int_\Omega \phi \int_0^{s_1} f(\xi)\,d\xi dx \\
&+ \int_\Omega \mathbf{K}M_1(s_1)\nabla p_1 \cdot \nabla p_1\,dx + \int_\Omega \mathbf{K}M_2(s_2)\nabla p_2 \cdot \nabla p_2\,dx \\
&= \int_\Omega \mathbf{K}\rho_1(p_1)\mathbf{g} \cdot M_1(s_1)\nabla p_1\,dx + \int_\Omega \mathbf{K}\rho_2(p_2)\mathbf{g} \cdot M_2(s_2)\nabla p_2\,dx \\
&- \int_\Omega \rho_1(p_1)s_1 f_p g_1(p_1)\,dx - \int_\Omega \rho_2(p_2)s_2 f_p g_2(p_2)\,dx \\
&+ \int_\Omega \rho_1(p_1)s_1^I f_I g_1(p_1)\,dx + \int_\Omega \rho_2(p_2)s_2^I f_I g_2(p_2)\,dx.
\end{aligned}$$

Main estimate

We have now

$$\int_{Q_T} M_1(s_1)\nabla p_1 \cdot \nabla p_1 \, dxdt + \int_{Q_T} M_2(s_2)\nabla p_2 \cdot \nabla p_2 \, dxdt \\ \leq C_1(1 + ||p_1||_{L^2(Q_T)} + ||p_2||_{L^2(Q_T)}) \qquad (4)$$

but, $p_1 = p - \bar{p}(s_1)$, $p_2 = p - \tilde{p}(s_1)$, then

$$\int_{Q_T} M(s_1)|\nabla p|^2 + \int_{Q_T} |\nabla \beta(s_1)|^2 = \int_{Q_T} M_1(s_1)|\nabla p_1|^2 + \int_{Q_T} M_2(s_2)|\nabla p_2|^2 \leq C_2$$

∇p , $\nabla \beta(s_1)$ and $\sqrt{M_i}\nabla p_i$ are bounded in $(L^2(Q_T))^N$.

Consequently, we show that

$\partial_t(\rho_i(p_i)s_i)$ is bounded in $L^2(0, T; (H^1_{\Gamma_1}(\Omega))')$

which lead to compactness result.

Physically relevant assumptions

(H1) $0 < \phi_0 \leq \phi(x) \leq \phi_1$ a.e. $x \in \Omega$.

(H2)

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty \quad \text{and} \quad (\mathbf{K}(x)\xi, \xi) \geq k_0|\xi|^2.$$

(H3) M_1 , M_2 $\in \mathcal{C}^0([0,1];\mathbb{R}^+)$, $M_1(0) = 0$, $M_2(0) = 0$,

$$M_1(s_1) + M_2(s_2) \geq m_0 > 0.$$

(H4) $\alpha = \frac{M_1 M_2}{M_1 + M_2}\frac{df}{ds} \in \mathcal{C}^1([0,1];^+)$, $\beta(s) := \int_0^s \alpha(z)dz$
β^{-1} is an Hölder function of order θ , with $0 < \theta \leq 1$.

(H5) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ a.e. $(t,x) \in Q_T$

(H6) $\rho_i \in \mathcal{C}^2(\mathbb{R})$ increasing, $0 < \rho_m \leq \rho_i(p_i) \leq \rho_M$.

(H7) Capillary pressure function $f \in \mathcal{C}^1([0,1];\mathbb{R}^-)$

$$0 < \underline{f} \leq \frac{df}{ds}.$$

Cauchy problem

Theorem 1. *Let (H1)-(H7) hold. Let* (p_1^0, p_2^0) *belongs to* $L^2(\Omega)^2$. *Then there exists* (p_1, p_2) *satisfying*

$$p_i \in L^2(Q_T), \sqrt{M_i}\nabla p_i \in (L^2(Q_T))^N,$$

$$\phi\partial_t(\rho_i(p_i)s_i) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),$$

$$0 \leq s_i(t,x) \leq 1 \textit{ a.e in } Q_T, \ \beta(s_1) \in L^2(0,T;H^1(\Omega))$$

such that for all $\varphi_i, \in C^1(0,T;H^1_{\Gamma_1}(\Omega))$ *with* $\varphi_i(T) = 0$,

$$-\int_{Q_T} \phi\rho_i(p_i)s_i\partial_t\varphi_i\, dxdt - \int_\Omega \phi(x)\rho_i(p_i^0(x))s_i^0(x)\varphi_i(0,x)\, dx$$

$$+\int_{Q_T} \mathbf{K}M_i(s_i)\rho_i(p_i)\nabla p_i\cdot\nabla\varphi_i\, dxdt - \int_{Q_T} \mathbf{K}M_i(s_i)\rho_i^2(p_i)\mathbf{g}\cdot\nabla\varphi_i\, dxdt$$

$$+\int_{Q_T} \rho_i(p_i)s_i f_P\varphi\, dxdt = \int_{Q_T} \rho_i(p_i)s_i^I f_I\varphi_i\, dxdt, \quad i = 1,2.$$

Finite Volume scheme for water-gas flow

The gas equation :

$$\phi\partial_t(\rho(p)s) - \mathrm{div}(\rho(p)M_1(s)\nabla p) - \mathrm{div}(\rho(p)\nabla\beta(s)) + \rho(p)sf_P = 0. \qquad (A)$$

The water equation :

$$\phi\partial_t s + \mathrm{div}(M_2(s)\nabla p) - \mathrm{div}(\nabla\beta(s)) + sf_P = f_P - f_I. \qquad (B)$$

s : saturation of gas, p : global pressure, $\beta(s)$: capillary term, $\rho(p)$: density of gas (increasing and bounded function).

Finite Volume scheme for water-gas flow

$$\phi\partial_t(\rho(p)s) - \mathrm{div}(\rho(p)M_1(s)\nabla p) - \mathrm{div}(\rho(p)\nabla\beta(s)) + \rho(p)sf_P = 0. \qquad (A)$$

$$\phi\partial_t s + \mathrm{div}(M_2(s)\nabla p) - \mathrm{div}(\nabla\beta(s)) + sf_P = f_P - f_I. \qquad (B)$$

- A priori estimate on pressure. Multiply (A) scalary by p + (B) scalary by $g(p) = -\int_0^p \rho(q)\,dq$, we get

$$\frac{d}{dt}\int_\Omega \phi s\mathcal{H}(p)\,dx + \int_\Omega \rho(p)M(s)|\nabla p|^2\,dx = \int_\Omega (f_P - f_I)g(p)\,dx - \int_\Omega \mathcal{H}(p)sf_P\,dx.$$

$$p\partial_t(\rho(p)s) + g(p)\partial_t s = \partial_t(s\mathcal{H}(p)), \qquad \text{and } \mathcal{H}(p) = \rho(p)p + g(p) \geq 0$$

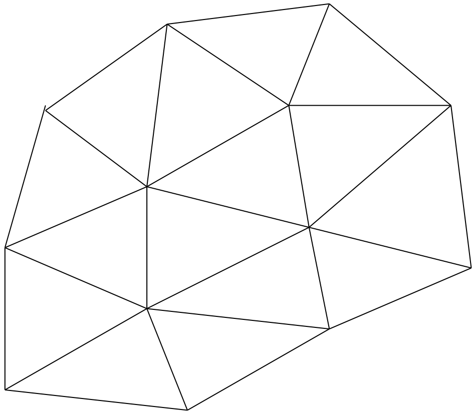
- A priori estimate on saturation. Multiply (B) scalary by $\beta(s)$, we get

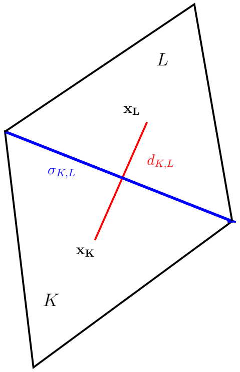
$$\frac{d}{dt}\int_\Omega \phi\Big(\int_0^s \beta(z)\,dz\Big)\,dx + \int_\Omega \nabla\beta(s)\cdot\nabla\beta(s)\,dx$$

$$= \int_\Omega M_2(s)\mathbf{K}\nabla p\cdot\nabla\beta(s)\,dx + \int_\Omega (1-s)(f_P - f_I)\beta(s)\,dx,$$

Finite Volume scheme for water-gas flow

- Discretization. Let Ω be an open bounded connected subset in R^d
  $(t^i)_{i=0,N}$ a subdivision of the interval $(0,T)$
  $\mathcal{T}$: admissible finite volume mesh of Ω (Eymard-Gallouët-Herbin).





- Notation.
  x_K the center of K
  $N(K)$ the set of the neighbors of K
  $d_{K,L} = d(x_K, x_L)$
  $\sigma_{K,L}$ the interface between K and L
  $\tau_{K|L} = \frac{|\sigma_{K,L}|}{d_{K,L}}$ transmissibility through $\sigma_{K,L}$

Formulation of scheme

- Initial condition:

$$p_K^0 = \frac{1}{|K|}\int_K p_0(x)\,dx, \qquad s_K^0 = \frac{1}{|K|}\int_K s_0(x)\,dx, \tag{4}$$

- Time derivative:

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \phi\partial_t(\rho(p)s)\;dxdt \approx |K|\,\phi_K\,\frac{\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n}{\Delta t}$$

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \phi\partial_t s\;dxdt \approx |K|\,\phi_K\,\frac{s_K^{n+1} - s_K^n}{\Delta t}$$

- Capillary terms :

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \Delta\beta(s) \approx \sum_{L\in N(K)} \tau_{K|L}\Big(\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big),$$

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \operatorname{div}(\rho(p)\nabla\beta(s)) \approx \sum_{L\in N(K)} \tau_{K,L}\,\rho_{K,L}^{n+1}\Big(\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big).$$

$\rho_{K,L} = \frac{1}{p_L - p_K}\int_{p_L}^{p_K} \rho(q)\,dq$.

Formulation of scheme

- Convective terms : Upwind scheme according to $-\nabla p \cdot \eta$ on interfaces.

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \operatorname{div}(M_2(s)\nabla p) \quad \approx \tau_{K,L}\begin{cases} M_2(s_K^{n+1})(p_L^{n+1}-p_K^{n+1}) & \text{if } p_L^{n+1}-p_K^{n+1}\leq 0 \\ M_2(s_L^{n+1})(p_L^{n+1}-p_K^{n+1}) & \text{if } p_L^{n+1}-p_K^{n+1} > 0, \end{cases}$$

$$\approx G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})$$

where $dp_{K,L}^{n+1} = \tau_{K,L}(p_L^{n+1}-p_K^{n+1})$, and $z = z^+ - z^-$, for all $z \in \mathbb{R}$.

In the same way

$$-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \operatorname{div}(\rho(p)M_1(s)\nabla p) \approx \rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})$$

$$G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) = -M_1(s_L^{n+1})(dp_{K,L}^{n+1})^+ + M_1(s_K^{n+1})(dp_{K,L}^{n+1})^-$$

Formulation of scheme

- Main properties on the numerical convection flux functions G_i

  Classical monotony and consistency properties :

  - $s \mapsto G_i(s,\cdot,\cdot)$ is non-decreasing, $s \mapsto G_i(\cdot,s,\cdot)$ is non-increasing
  - $G_1(s,s,q) = -M_1(s)\,q\,,\qquad G_2(s,s,q) = M_2(s)\,q$
  - $G_i(s_1,s_2,q) = -G_i(s_2,s_1,-q)$

  And for the considered system, we have : there exists a constant m_0 such that

  $$\Big(G_2(s_1,s_2,q) - G_1(s_1,s_2,q)\Big)q \geq m_0|q|^2, \qquad \text{for all } s_1,s_2,q \in \mathbb{R}.$$

  It is a key point to show the discret gradient of pressure is bounded.

- Gravity terms : upwind scheme according to $\mathbf{g}\cdot\eta$ on the interface $\sigma_{K,L}$.

Formulation of scheme

$$|K|\,\phi_K \frac{\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n}{\Delta t} - \sum_{L\in N(K)} \tau_{K,L}\rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))$$

$$+ \sum_{L\in N(K)} \rho_{K,L}^{n+1} G_{1,K,L}^{n+1} + F_{1,K,L}^{(n+1)} + |K|\,\rho(p_K^{n+1})s_K^{n+1} f_{P,K} = 0,$$

$$|K|\,\phi_K \frac{s_K^{n+1} - s_K^n}{\Delta t} - \sum_{L\in N(K)} \tau_{K,L}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))$$

$$+ \sum_{L\in N(K)} G_{1,K,L}^{n+1} + F_{2,K,L}^{(n+1)} + |K|\,(s_K^{n+1} - 1) f_{P,K} = -\,|K|\, f_{I,K}$$

where $G_{i,K,L}^{n+1}$ and $F_{i,K,L}^{n+1}$ denotes the approximation of convective and gravity terms, $i = 1, 2$.

A priori estimates

Lemma 1. *(Nonnegativity) Let* $(s_K^0)_{K\in\mathcal{T}} \in [0,1]$. *Then, the solution* $(s_K^n)_{K\in\mathcal{T},n\in\{0,\ldots,N\}}$ *, of the finite volume scheme remains in* $[0,1]$.

Proposition 1. *(Estimates on discret gradient pressure and saturation)*
Let $(p_K^n, s_K^n)_{K\in\mathcal{T},n\in\{0,\ldots,N\}}$ *, be a solution of the finite volume scheme. Then, there exist a constant* $C>0$, *depending on* Ω, T, s_0 *and* p_0 *such that*

$$\sum_{K\in\mathcal{T}} |K|\, s_K^N \mathcal{H}(p_K^N) - \sum_{K\in\mathcal{T}} |K|\, s_K^0 \mathcal{H}(p_K^0) + \\ + \frac{c_1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K,L}\left|p_K^{n+1}-p_L^{n+1}\right|^2 \le C$$

$$\sum_{K\in\mathcal{T}} |K|\, B(s_K^N) - \sum_{K\in\mathcal{T}} |K|\, B(s_K^0) + \\ + \frac{1}{4}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K,L}\left|\beta(s_K^{n+1})-\beta(s_L^{n+1})\right|^2 \le C$$

where $B(s)=\int_0^s \beta(r)\,dr$.

Sketch of the proof : Estimate on discret global pressure

Multiply the gas discrete equation by p_K^{n+1} and the water discrete equation by $g(p_K^{n+1}) = -\int_0^{p_K^{n+1}} \rho(q)\, dq$ and adding them :

- Evolution term

$$E_1 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K \Big((\rho(p_K^{n+1}) s_K^{n+1} - \rho(p_K^n) s_K^n) p_K^{n+1} + (s_K^{n+1} - s_K^n) g(p_K^{n+1}) \Big).$$

We have

$$E_1 \geq \sum_{K\in\mathcal{T}} |K|\, s_K^N \mathcal{H}(p_K^N) - \sum_{K\in\mathcal{T}} |K|\, s_K^0 \mathcal{H}(p_K^0),$$

where $\mathcal{H}(p) \geq 0$ and $|\mathcal{H}(p)| \leq c|p|$, for all p.

Sketch of the proof : Estimate on discret global pressure

Multiply the gas discrete equation by p_K^{n+1} and the water discrete equation by $g(p_K^{n+1}) = -\int_0^{p_K^{n+1}} \rho(q)\,dq$ and adding them : where $\mathcal{H}(p) \geq 0$ and $|\mathcal{H}(p)| \leq c|p|$, for all p.

- Capillary term. Let us denote

$$\sum_{n,\sigma} = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \tau_{K,L}.$$

Gathering by edges , we obtain

$$E_2 = \sum_{n,\sigma}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\Big(\rho_{K,L}^{n+1}(p_L^{n+1} - p_K^{n+1}) + (g(p_L^{n+1}) - g(p_K^{n+1}))\Big)$$

Choose now the density at interfaces as $\rho_{K,L} = \dfrac{1}{p_L - p_K}\displaystyle\int_{p_L}^{p_K} \rho(q)\,dq$ so,

$\rho_{K,L}(p_L - p_K) = -(g(p_L) - g(p_K))$; this leads to

$$E_2 = 0. \tag{4}$$

Sketch of the proof : Estimate on discret global pressure

Multiply the gas discrete equation by p_K^{n+1} and the water discrete equation by $g(p_K^{n+1}) = -\int_0^{p_K^{n+1}} \rho(q)\,dq$ and adding them : where $\mathcal{H}(p) \geq 0$ and $|\mathcal{H}(p)| \leq c|p|$, for all p.

- Global pressure term. Gathering by edges and using the fact that the fluxes are conservative, we obtain

$$E_3 = -\sum_{n,\sigma} \Big(\rho_{K,L}^{n+1}(p_L^{n+1} - p_K^{n+1})G_{1,K,L}^{n+1} + (g(p_L^{n+1}) - g(p_K^{n+1}))G_{2,K,L}^{n+1} \Big)$$

Using again $\rho_{K,L}(p_L - p_K) = -(g(p_L) - g(p_K))$,

$$E_3 = \sum_{n,\sigma} \rho_{K,L}^{n+1}(G_{2,K,L}^{n+1} - G_{1,K,L}^{n+1})(p_L^{n+1} - p_K^{n+1}),$$

and from the property of $\Big(G_2(s_1, s_2, q) - G_1(s_1, s_2, q)\Big)q \geq m_0|q|^2$, we deduce that

$$m_0\rho_m \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K,L} \left| p_K^{n+1} - p_L^{n+1} \right|^2 \leq E_3. \qquad (4)$$

Sketch of the proof : Estimate on discret saturation

We multiply the water discrete equation by $\beta(s_K^{n+1})$ then summing the resulting equation over K and n :

$$E_1 + E_2 + E_3 + E_4 = 0,$$

- evolutive term

$$E_1 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\,\phi_K (s_K^{n+1} - s_K^n)\beta(s_K^{n+1})$$

$$\geq \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\,\phi_K (B(s_K^{n+1}) - B(s_K^n)) \qquad \text{(since } B \text{ is convex).}$$

- dissipative term

$$E_2 = \frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \tau_{K,L}(\beta(s_L^{n+1}) - \beta(s_K^{n+1})^2.$$

- pressure term

$$E_3 = \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\beta(s_K^{n+1}).$$

Space and time translation estimates

Define the discret functions :

$$s_h \text{ and } u_h = \rho(p_h) s_h B(s_h)$$

constant per cylinder $(t^n, t^{n+1}] \times K$. We derive estimates on differences of space and time translates of the functions

$$\tilde{u}_h \text{ and } \tilde{s}_h$$

piecewise constant in t and constant in x for all K .
The sequences $\tilde{u}_h$ and $\tilde{s}_h$ are relatively compact in $L^1(Q_T)$.

Lemma 2. *There exists positive a constant $C > 0$ depending on Ω , T , u_0 and v_0 such that*

$$\int_{\Omega' \times (0,T)} \left| \bar{U}(t, x+y) - \bar{U}(t,x) \right|^2 \, dx\, dt \leq C\, |y| \, (|y| + 2h), \tag{5}$$

for all $y \in \mathbb{R}^3$ with $\Omega' = \{x \in \Omega,\ [x, x+y] \subset \Omega\}$, and

$$\iint_{\Omega \times (0,T-\tau)} \left| \bar{U}(t+\tau, x) - \bar{U}(t,x) \right|^2 \, dx\, dt \leq C(\tau + \Delta t), \tag{6}$$

for all $\tau \in (0,T)$.

Convergence of FV scheme

Théorème Assume that $p_0 \in L^2(\Omega)$ and $0 \leq s_0 \leq 1$. Let (s_h, p_h) be the discrete solution generated by the finite volume scheme. Then, as $h \longrightarrow 0$, (s_h, p_h) converges to (s,p) :

$$0 \leq s \leq 1, \ \beta(s) \in L^2(0,T; H^1_{\Gamma_w}(\Omega)),$$
$$p \in L^2(0,T; H^1_{\Gamma_w}(\Omega)),$$

such that for all $\varphi, \xi \in \mathcal{C}^1([0,T] \times \Omega))$, $\varphi(T,\cdot) = \xi(T,\cdot) = 0$

$$-\int_{Q_T} \phi\rho(p)s)\partial_t\varphi \, dxdt - \int_\Omega \rho(p_0)s_0\phi(0,x)\, dx + \int_{Q_T} \rho(p)M_1(s)\nabla p \cdot \nabla\varphi \, dxdt$$
$$+ \int_{Q_T} \rho(p)\nabla\beta(s) \cdot \nabla\varphi \, dxdt + \int_{Q_T} \rho(p)sf_P\varphi \, dxdt = 0,$$

$$-\int_{Q_T} \phi s\partial_t\xi \, dxdt - \int_\Omega s_0\xi(0,x)\, dx + \int_{Q_T} \nabla\beta(s) \cdot \nabla\xi \, dxdt$$
$$-\int_{Q_T} M_2(s)\nabla p \cdot \nabla\xi \, dxdt + \int_{Q_T} sf_P\xi \, dxdt = \int_{Q_T} (f_P - f_I)\xi \, dxdt.$$