Discontinuous Galerkin approximation of two-phase flows in heterogeneous porous media with distinct capillary pressures

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Introduction

- We consider two-phase, immiscible, incompressible flows through isothermal and indeformable porous media
- The present results fit more directly into the context of secondary oil recovery and oil trapping effects
- They constitute (possibly) a first step towards water-hydrogen flows around underground repositories

Discontinuous Galerkin (dG) methods

- dG methods can be viewed as
  - FE-based methods using piecewise polynomials discontinuous across mesh elements
  - FV-based high-order methods using numerical fluxes
- dG methods were introduced in the early 70's for neutron transport
- They experienced a vigorous development over the last decade
  - for two-phase flows, see Bastian (99), Bastian & Rivière (03), Eslinger (05), Klieber & Rivière (06), Epshteyn & Rivière (07)
- Attractive features include
  - local conservation properties
  - flexibility (non-matching grids, variable polynomial degrees)
  - ability to capture shocks sharply

Heterogeneous porous media

- Distinct capillary pressure curves in adjacent subdomains lead to discontinuous saturations and more generally to nonlinear interface conditions, see Bear (78)
- Existence of solutions, see Amaziane, Bourgeat & El Amri (96)
- FV methods, see Enchéry, Eymard & Michel (06), Cancès (09), Cancès, Gallouët & Porretta (09)
- Current dG methods cannot handle heterogeneous two-phase flows with distinct capillary pressure curves

Objectives

- Formulate a robust dG method for such flows by revisiting the penalty strategy in the design of the method
- Three key ingredients
  - accurate (total) velocity reconstruction from pressure gradients using Raviart-Thomas FE (at no extra cost)
  - weighted averages in the consistency terms and harmonic averages in the penalties
  - weak enforcement of nonlinear interface conditions

Outline

- Model formulation
- DG approximation
- Numerical results

Model formulation

- Governing equations
- Fractional flow formulation
- Interface conditions

Governing equations

- Mass conservation for each phase (incompressible, immiscible)

$$\partial_t(\phi S_\alpha) + \nabla\cdot u_\alpha = q_\alpha \qquad \alpha \in \{n, w\}$$

  ϕ: (constant) porosity, S_α: phase saturation, u_α: phase velocity, q_α: source/sink
- $S_n + S_w = 1$, $S := S_n \in [S_{nr}, 1 - S_{wr}]$
- Generalized Darcy's law (no gravity)

$$u_\alpha = -K\lambda_\alpha(S)\nabla p_\alpha$$

  K: absolute permeability, λ_α: phase mobility, p_α: phase pressure
- Capillary pressure

$$p_n - p_w = \pi(S)$$

Fractional flow formulation

- Total mobility $\lambda = \lambda_w + \lambda_n$, fractional flow $f = \lambda_n/\lambda$
- Global pressure (Chavant & Jaffré 86)

$$p = p_w + \int_{S_{nr}}^{S} f(\sigma)\pi'(\sigma)d\sigma + \pi(S_{nr}) = p_n + \int_{S_{nr}}^{S} (f(\sigma) - 1)\pi'(\sigma)d\sigma$$

- Total velocity $u = u_w + u_n$ s.t.

$$\nabla\cdot u = q_w + q_n \qquad u = -\lambda K \nabla p$$

- Non-wetting phase mass conservation becomes

$$\phi\partial_t S + \nabla\cdot(uf(S)) - \nabla\cdot(\epsilon(S)\pi'(S)\nabla S) = q_n$$

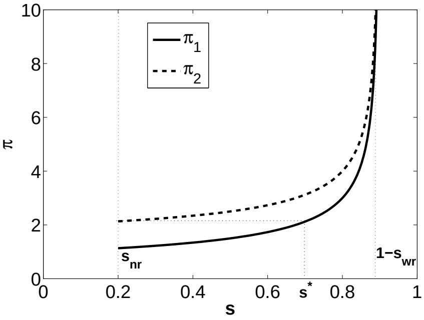
with $\epsilon(S) := \lambda_w(S)f(S)K$

Fractional flow formulation

- Sequential approach to march in time: For $m = 0, 1, \ldots$
  1. solve elliptic equation for global pressure
  $$\nabla\cdot(\lambda(S^m)K\nabla p^{m+1}) = q_w^{m+1} + q_n^{m+1}$$
  2. reconstruct total velocity
  $$u^{m+1} = -\lambda(S^m)K\nabla p^{m+1}$$
  3. advance in time saturation equation (implicit Euler)
  $$\phi\frac{S^{m+1}-S^m}{\tau^m} + \nabla\cdot(u^{m+1}f(S^{m+1})) - \nabla\cdot(\epsilon(S^m)\pi'(S^m)\nabla S^{m+1}) = q_n^{m+1}$$
- S^0 given by IC
- BC's can be of Dirichlet or Neumann type for both pressure and saturation

Interface conditions I

- For simplicity, two subdomains Ω_β, $\beta \in \{1,2\}$, with different rock properties, Ω_1 being upwind sudomain w.r. to total velocity
- Up to rescaling, both S_β's take values in $[S_{nr}, 1 - S_{wr}]$
- Assumption on the capillary pressure curves



- Critical value $S^* = \pi_1^{-1}\pi_2(S_{nr})$

Interface conditions II

- Jump $[\![a]\!] := a_1 - a_2$
- Interface conditions on the saturation
  - flux continuity $[\![uf(S) - \epsilon(S)\pi'(S)\nabla S]\!]\cdot n_\Gamma = 0$
  - $S_2 = S_{nr}$ if $S_1 \in [S_{nr}, S^*]$
- Interface conditions on the pressure
  - flux continuity $[\![-\lambda K \nabla p]\!]\cdot n_\Gamma = 0$
  - continuity of (some) phase pressures

$$
\begin{aligned}
[\![p_w]\!] &= 0 \qquad \text{if } S_1 \in [S_{nr}, S^*] \\
[\![p_w]\!] = [\![p_n]\!] &= 0 \qquad \text{if } S_1 \in [S^*, 1 - S_{wr})
\end{aligned}
$$

Interface conditions III

- dG methods can weakly enforce the values of interface jumps
- Reformulate interface cdtn. on saturation as $[\![S]\!] = J(S_1)$ with

$$J(S) = \begin{cases} S & \text{if } S_1 \in [S_{nr}, S^*] \\ S - \pi_2^{-1}(\pi_1(S)) & \text{if } S_1 \in [S^*, 1 - S_{wr}) \end{cases}$$

- Reformulate interface cdtn. on pressure as $[\![p]\!] = G(S)$ with

$$G(S) = \begin{cases} \int_{S_{nr}}^{S_1} f_1(\sigma)\pi_1'(\sigma)d\sigma + [\![\pi(S_{nr})]\!] & \text{if } S_1 \in [S_{nr}, S^*] \\ [\![\int_{S_{nr}}^{S} (f(\sigma) - 1)\,\pi'(\sigma)d\sigma]\!] & \text{if } S_1 \in [S^*, 1 - S_{wr}) \end{cases}$$

- For $S_1 \in [S^*, 1 - S_{wr})$, this yields $[\![p_n]\!] = 0$ and $[\![\pi]\!] = 0$ so that $[\![p_w]\!] = 0$

dG approximation

- Discrete setting
- Pressure equation
- Velocity reconstruction
- Saturation equation

Discrete setting

- Discrete times $\{t^m\}_{0\leq m\leq N}$ with time steps $\tau^m := t^{m+1} - t^m$
- Shape-regular meshes $\{\mathcal{T}_h\}_{h>0}$ (kept fixed in time)
- Generic element $T \in \mathcal{T}_h$ with diameter h_T, meshsize $h := \max_{T\in\mathcal{T}_h} h_T$
- Discrete pressures and saturations belong to the same dG space

$$V_h^k := \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_k(T)\}$$

with polynomial degree $k \geq 1$

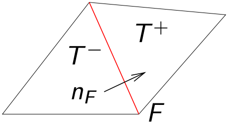
Mesh faces

- Meshes are fitted to the partition of Ω into $\Omega_1 \cup \Omega_2$
- Interfaces collected in $\mathcal{F}_h^i$, boundary faces in $\mathcal{F}_h^\partial$, mesh faces in $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$
- Interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ is exactly covered by faces collected in $\mathcal{F}_h^\Gamma$
- Remaining interfaces $\mathcal{F}_h^{i*} := \mathcal{F}_h^i \setminus \mathcal{F}_h^\Gamma$

Weighted averages and jumps

- Diffusion-weighted averages in dG methods have been introduced by Burman & Zunino (06) and AE, Stephansen & Zunino (09); they are important for locally vanishing diffusion
- Diffusion coefficient a two-valued on interface $F = \partial T^- \cap \partial T^+$ leads to weights

$$\omega_{T^-,F} := \frac{a_{T^+,F}}{a_{T^-,F}+a_{T^+,F}} \qquad \omega_{T^+,F} := \frac{a_{T^-,F}}{a_{T^-,F}+a_{T^+,F}}$$



- For two-valued function v on F with traces $v^\pm$, define its weighted average and jump as

$$\{v\}_\omega := \omega_{T^-,F} v^- + \omega_{T^+,F} v^+ \qquad [\![v]\!] = v^- - v^+$$

$\omega_{T^\pm,F} = 1/2$ leads to usual arithmetic average (homogeneous case)

Pressure equation I

- Find $p_h^{m+1} \in V_h^k$ s.t. for all $z_h \in V_h^k$ (only Dirichlet BC's)

$$\sum_{T\in\mathcal{T}_h} \int_T -(\nabla\cdot(\lambda(S_h^m)K\nabla p_h^{m+1}) + q_w^{m+1} + q_n^{m+1})z_h$$

$$+ \sum_{F\in\mathcal{F}_h^i} \int_F n_F\cdot[\![\lambda(S_h^m)K\nabla p_h^{m+1}]\!]\{z_h\}_{\bar{\omega}}$$

$$+ \sum_{F\in\mathcal{F}_h} \int_F [\![p_h^{m+1}]\!]' \left(-n_F\cdot\{\lambda(S_h^m)K\nabla z_h\}_\omega + \gamma_F \frac{\sigma_F}{h_F}[\![z_h]\!]\right) = 0$$

where

$$[\![p_h^{m+1}]\!]' = \begin{cases} [\![p_h^{m+1}]\!] & \text{if } F \in \mathcal{F}_h^{i*} \\ [\![p_h^{m+1}]\!] - G(S_h^m) & \text{if } F \in \mathcal{F}_h^{\Gamma} \\ p_h^{m+1} - p_D & \text{if } F \in \mathcal{F}_h^{\partial} \end{cases}$$

Pressure equation II

- Reference diffusion $a_{T^\pm,F} = \|(\lambda(S_h^m)K)|_{T^\pm}\|_{L^\infty(F)}$
- Penalty coefficient based on harmonic average of diffusivity

$$\gamma_F = \langle a \rangle_F := \frac{2a_{T^-,F}a_{T^+,F}}{a_{T^-,F} + a_{T^+,F}}$$

  cf. AE, Stephansen & Zunino (09)

- User-dependent parameter σ_F depending on mesh regularity (and k)

Pressure equation III

- The above method is the Symmetric (Weighted) Interior Penalty dG method

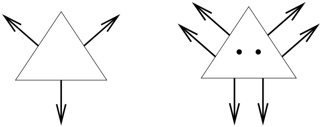
$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T \lambda(S_h^m)K\nabla p_h^{m+1}\cdot\nabla z_h \\
&-\sum_{F\in\mathcal{F}_h}\int_F \left(n_F\cdot\{\lambda(S_h^m)K\nabla p_h^{m+1}\}_\omega [\![z_h]\!] + n_F\cdot\{\lambda(S_h^m)K\nabla z_h\}_\omega [\![p_h^{m+1}]\!]\right) \\
&+\sum_{F\in\mathcal{F}_h}\int_F \gamma_F\frac{\sigma_F}{h_F}[\![p_h^{m+1}]\!][\![z_h]\!] = \sum_{T\in\mathcal{T}_h}\int_T \left(q_w^{m+1}+q_n^{m+1}\right) z_h \\
&+\sum_{F\in\mathcal{F}_h^\Gamma}\int_F \left(-n_F\cdot\{\lambda(S_h^m)K\nabla z_h\}_\omega + \gamma_F\frac{\sigma_F}{h_F}[\![z_h]\!]\right) G(S_h^m) \\
&+\sum_{F\in\mathcal{F}_h^\partial}\int_F \left(-n_F\cdot\lambda(S_h^m)K\nabla z_h + \gamma_F\frac{\sigma_F}{h_F}z_h\right) p_D,
\end{aligned}
$$

Velocity reconstruction I

- In dG methods, a locally conservative velocity field can be postprocessed at no extra cost
  - Kim (07), AE, Nicaise & Vohralík (07)
- Raviart–Thomas(–Nédélec) finite element space of degree l, $l \in \{k-1, k\}$

$$\mathbf{RT}_l(\mathcal{T}_h) := \{u_h \in \mathbf{H}(\mathrm{div});\ \forall T \in \mathcal{T}_h,\ u_h|_T \in [\mathbb{P}_l(T)]^d + x\mathbb{P}_l(T)\}$$

- Local degrees of freedom for $l = 0$ and $l = 1$



Velocity reconstruction II

- Local prescription of degrees of freedom for u_h^{m+1} (on matching simplicial meshes)
  - For all $F \in \mathcal{F}_h$ with $F \subset \partial T$ and for all $q \in \mathbb{P}_l(F)$

$$\int_F (u_h^{m+1} \cdot n_F) q = \int_F \left(-n_F \cdot \{\lambda(S_h^m) K \nabla p_h^{m+1}\}_\omega + \gamma_F \frac{\sigma_F}{h_F} [\![p_h^{m+1}]\!]' \right) q$$

  - For all $r \in [\mathbb{P}_{l-1}(T)]^d$

$$\int_T u_h^{m+1} \cdot r = -\int_T \lambda(S_h^m) K \nabla p_h^{m+1} \cdot r + \sum_{F \subset \partial T} \int_F \omega_{T,F} \lambda(S_h^m) n_F \cdot K \cdot r [\![p_h^{m+1}]\!]'$$

- Key properties are
  - local conservativity
  - spectrally optimal divergence

$$\int_T (\nabla \cdot u_h^{m+1}) \xi = \int_T (q_w^{m+1} + q_n^{m+1}) \xi \qquad \forall \xi \in \mathbb{P}_l(T)$$

Saturation equation I

- Implicit Euler and semi-linearization of diffusive term

$$\phi\frac{S^{m+1}-S^m}{\tau^m} + \nabla\cdot(u^{m+1}f(S^{m+1})) - \nabla\cdot(\epsilon(S^m)\pi'(S^m)\nabla S^{m+1}) = q_n^{m+1}$$

- Symmetric (Weighted) IP dG for diffusion
- Upwind for advection by total velocity
- Reference diffusion $a_{T^\pm,F} = \|(\epsilon(S_h^m)\pi'(S_h^m))|_{T^\pm}\|_{L^\infty(F)}$
- Penalty coefficient δ_F based on harmonic average of a

Saturation equation II

- Find $S_h^{m+1} \in V_h^k$ s.t. for all $v_h \in V_h^k$ (only Dirichlet BC's)

$$\sum_{T\in\mathcal{T}_h}\int_T \left(\phi\frac{S_h^{m+1}-S_h^m}{\tau^m} + \nabla\cdot(u_h^{n+1}f(S_h^{m+1}) - \epsilon(S_h^m)\pi'(s_h^m)\nabla S_h^{m+1}) - q_n^{m+1}\right)v_h$$

$$+\sum_{F\in\mathcal{F}_h^i}\int_F n_F\cdot[\![-u_h^{n+1}f(S_h^{m+1}) + \epsilon(S_h^m)\pi'(s_h^m)\nabla S_h^{m+1}]\!]\{v_h\}_{\bar{\omega}}$$

$$+\sum_{F\in\mathcal{F}_h}\int_F [\![S_h^{m+1}]\!]'\left(-n_F\cdot\{\epsilon(S_h^m)\pi'(S_h^m)\nabla v_h\}_\omega + \delta_F\frac{\sigma_F}{h_F}[\![v_h]\!]\right)$$

$$+\sum_{F\in\mathcal{F}_h^{i*}}\int_F \omega_{T\downarrow,F}|u_h^{n+1}\cdot n_F|[\![v_h]\!][\![f(S_h^{m+1})]\!] + \sum_{F\in\mathcal{F}_h^{\partial}}\int_F |u_h^{n+1}\cdot n_F|[\![v_h]\!][\![S_h^{m+1}]\!]' = 0$$

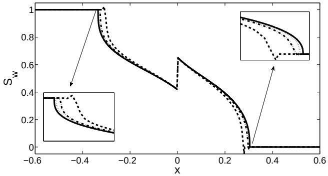
where

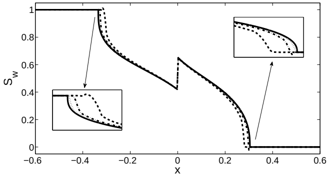
$$[\![S_h^{m+1}]\!]' = \begin{cases} [\![S_h^{m+1}]\!] & \text{if } F\in\mathcal{F}_h^{i*} \\ [\![S_h^{m+1}]\!] - J(S_{h1}^m) & \text{if } F\in\mathcal{F}_h^{\Gamma} \\ S_h^{m+1} - S_D & \text{if } F\in\mathcal{F}_h^{\partial} \end{cases}$$

Numerical results I

- Verification on test case with analytical (self-similar) solution (Van Duijn & Neef 98)
- $\phi = 1,\ S_{nr} = S_{wr} = 0$
- Brooks-Corey model for mobilities with parameter $\theta = 2$
- Absolute permeabilities $K_1 = 1$ and $K_2 = 0.64$
- Capillary pressure curves $\pi_\beta(S) = K_\beta^{-1/2}(1-S)^{-1/2}$
- No sources/sinks
- $\Omega = (-0.6, 0.6)$ with interface at $x = 0$

Numerical results II





- Exact (continuous line) and numerical solutions for approximation order $p = 1$ (left), $p = 2$ (right) and for different meshsizes
  - $h^{-1} = 128$, $\tau = 1.25 \times 10^{-2}$ (dashed line)
  - $h^{-1} = 256$, $\tau = 3.125 \times 10^{-3}$ (dashed-point line)
  - $h^{-1} = 512$, $\tau = 7.8125 \times 10^{-4}$ (point line)
- No limiters were used

Numerical results III

- Test case for pushing a blob of oil
- $\phi = 0.2$, $S_{nr} = S_{wr} = 0$
- Brooks-Corey model for mobilities with parameter $\theta = 2$
- Absolute permeabilities $K_1 = 1$ and $K_2 = 0.1$
- Capillary pressure curves ($S^* = 5^{-1/2} \simeq 0.4472$)

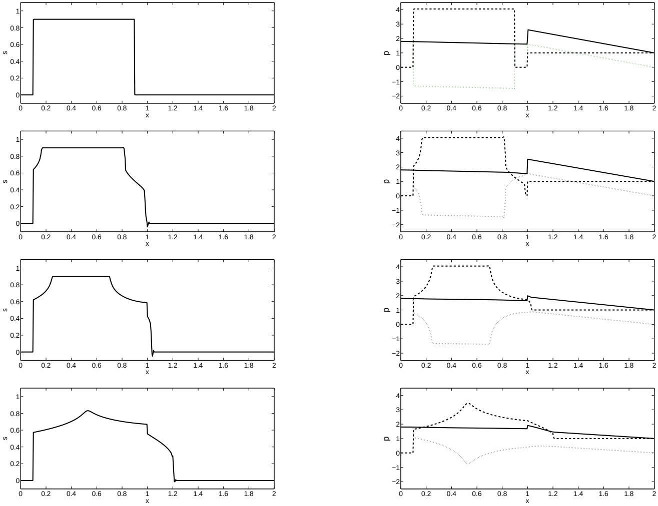
$$\pi_1(s) = 5s^2 \qquad \pi_2(s) = 4s^2 + 1$$

- No sources/sinks

Numerical results IV

- 1D setting with $\Omega_1 = (0,1)$ and $\Omega_2 = (1,2)$
- Dirichlet BC's on the pressure: $p|_{x=0} = 1.8$ and $p|_{x=2} = 1.0$
- Mixed BC's on saturation: $S|_{x=0} = 0$ and $\epsilon(S)\pi'(S)\frac{dS}{dx}|_{x=2} = 0$
- Discretization parameters: $k = 1$, $l = 1$, $h = 1/80$, $\tau = 0.001$
- No limiters were used

Case 1: Large enough IC for S



Case 2: Small IC for S