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DIOPTRE: A nine point cell centered scheme for anisotropic heterogeneous diffusion problems

R. Eymard, in collaboration with

L. Agelas and R. Herbin

IFP, Paris-Est and Provence universities

Properties of DIOPTRE scheme

It applies to anisotropic heterogeneous diffusion operators on general 2D polygonal grid

It provides conservative fluxes between control volumes

It is exact on any mesh if continuous solution is piecewise affine

It uses cell values as primary variables with stencil: 9-point in 2D, 27-point in 3D

It ensures convergence and symmetry properties

Principles of the scheme

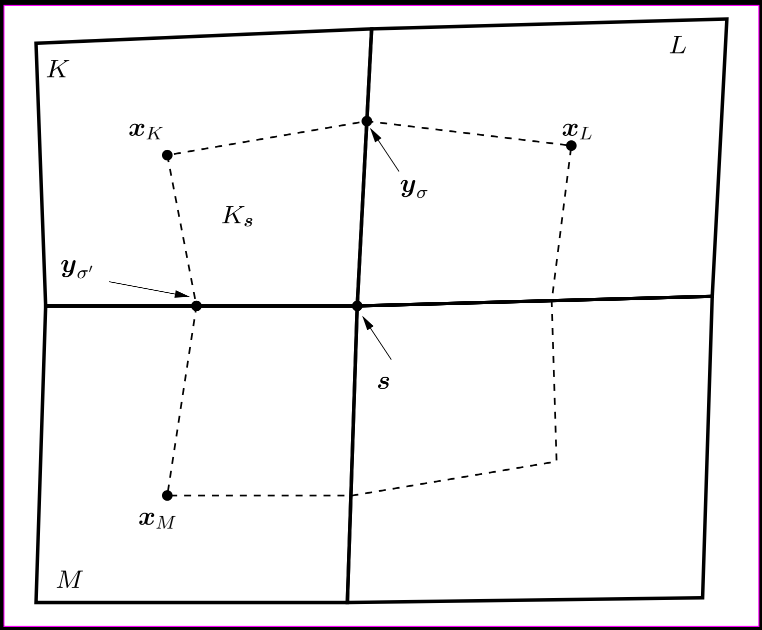
Define a discrete gradient $\nabla_{\mathcal{D}} u$ depending on cell and interfaces values

Define a discrete space $H_{\mathcal{D}}$ defining some of the interfaces values by given linear combinations

Define the scheme by $u \in H_{\mathcal{D}},\ \forall v \in H_{\mathcal{D}},\ \int_\Omega \Lambda \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v \mathrm{d}x = \int_\Omega f v \mathrm{d}x$, up to stabilization

Eliminate remaining interface values using flux conservation, as in O-scheme

First step: definition of discrete gradient



$$\mathcal{V} = \{K_s, K \in \mathcal{M}, s \in S_K\}$$

values $u^{\epsilon}_{K,s}$ at centers of edges

$$\epsilon = [x_K, y_{\sigma'}][y_{\sigma'}, s][s, y_\sigma][x_K, y_\sigma]$$

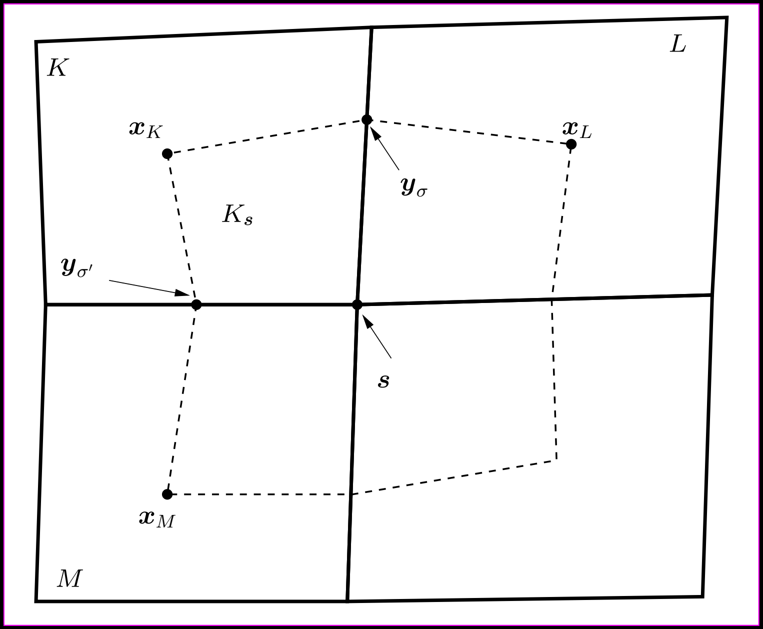
used in
$$|K_s| \nabla_{K,s} u = \sum_{\epsilon \in \mathcal{E}_{K,s}} |\epsilon| (u^{\epsilon}_{K,s} - u_K) n^{\epsilon}_{K,s}$$

Properties of such a discrete gradient:

weak cv of $\nabla_{\mathcal{D}} u_{\mathcal{D}}$ to ∇u in $L^2(\Omega)^d$
if $u_{\mathcal{D}}$ cv to u with some discrete norm bounded

cv of $\nabla_{\mathcal{D}} P_{\mathcal{D}} \varphi$ to $\nabla \varphi$ in $L^2(\Omega)^d$

Second step: determination of interface unknowns



1- Use continuity of flux (cf. MPFA)

elimination of $u^{\epsilon}_{K,s}$ for $\epsilon = [y_{\sigma'}, s][s, y_\sigma]$

provides

$u^{\epsilon}_{K,s}$ lin.comb. of four u_L

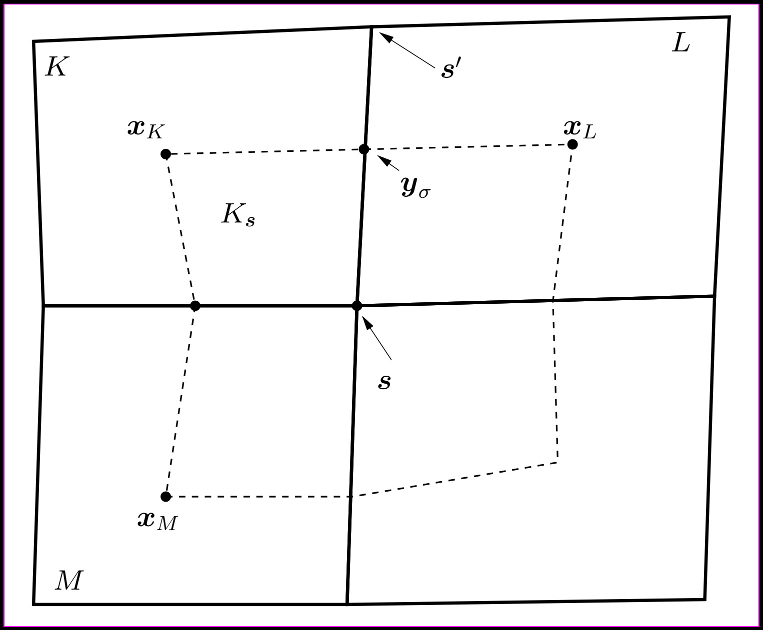
and of eight $u^{\epsilon'}_{K,s}$ for $\epsilon' = [x_K, y_\sigma] \ldots$

2- Define values by linear combination (cf. SUSHI)

$$u^{\epsilon'}_{K,s} = \frac{1}{2}(u_K + u_\sigma)$$

9-point stencil needs $u_\sigma = \alpha u_K + (1-\alpha) u_L$ ...

Averaging on edges



first idea:

$$y_\sigma = [x_K, x_L] \cap [s, s']$$

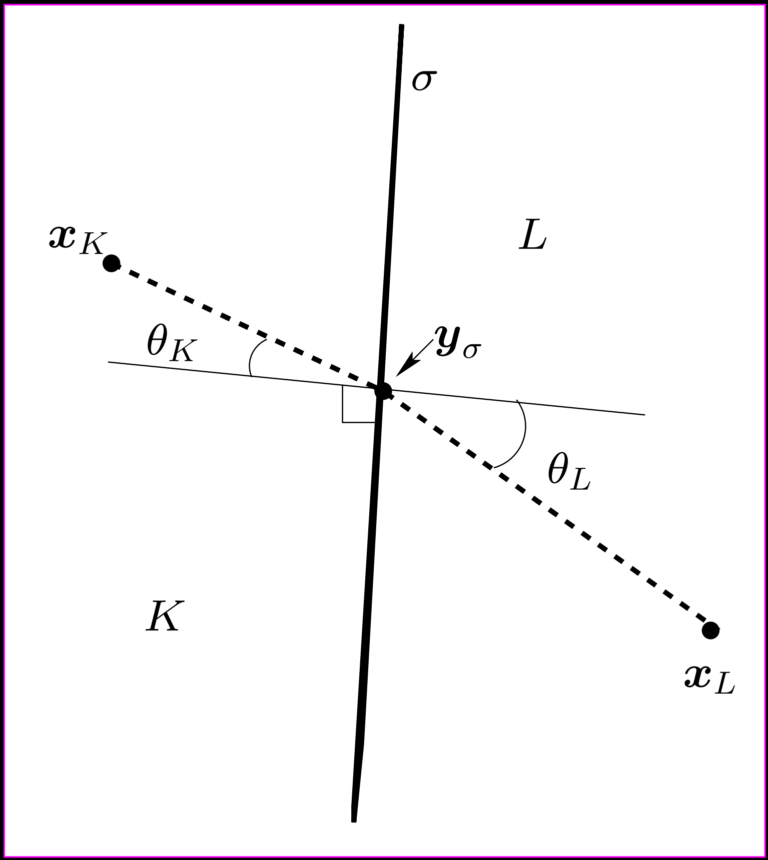
then barycentric expression

does not provide exact discrete solution
if continuous solution piecewise affine

second idea:
use "harmonic averaging" instead
of barycentric value...

Need two-point expression in y_σ

A problem which looks like optics...



Problem: find point $\boldsymbol{y}_\sigma$

s.t., for all u affine in K and L

continuous at the interface

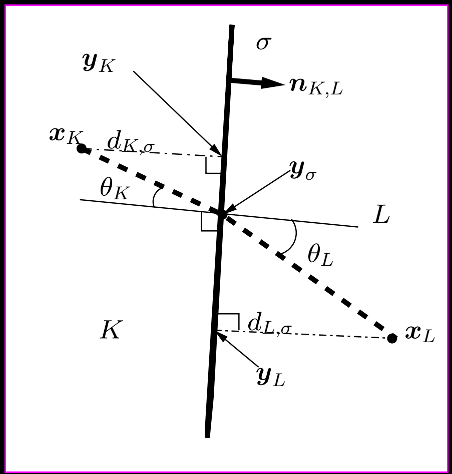
with $\boldsymbol{\Lambda_K \nabla u_{|K} \cdot n_{KL} = \Lambda_L \nabla u_{|L} \cdot n_{KL}}$

then $u(\boldsymbol{y}_\sigma)$ **given by lin. comb. of** $u(\boldsymbol{x}_K)$ **and** $u(\boldsymbol{x}_L)$

Recall: Snell-Descartes law

$$\boldsymbol{n_K \sin\theta_K = n_L \sin\theta_L}$$

Harmonic averaging points....



$$y_\sigma = \frac{\lambda_L d_{K,\sigma} y_L + \lambda_K d_{L,\sigma} y_K}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} + \frac{d_{K,\sigma} d_{L,\sigma}}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} (\lambda_K^\sigma - \lambda_L^\sigma)$$

with

$$\lambda_K = n_{KL} \cdot \Lambda_K n_{KL} \quad \lambda_K^\sigma = (\Lambda_K - \lambda_K \mathrm{Id}) n_{KL}$$

$$\lambda_L = n_{KL} \cdot \Lambda_L n_{KL} \quad \lambda_L^\sigma = (\Lambda_L - \lambda_L \mathrm{Id}) n_{KL}$$

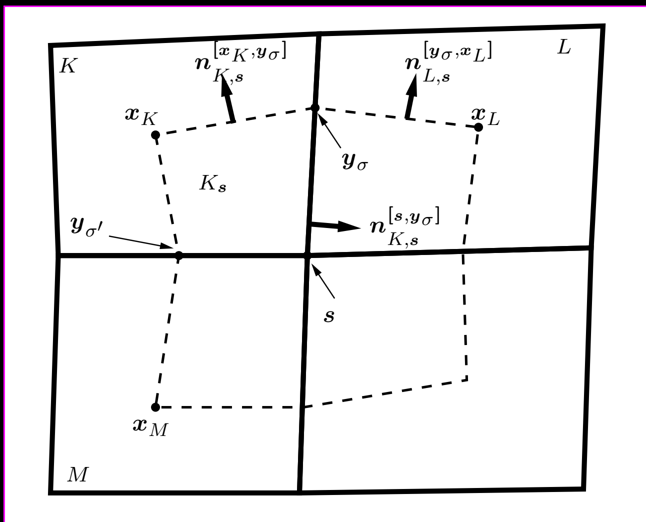
for all continuous function u affine in K and L s.t. $\Lambda_K \nabla u_{|K} \cdot n_{KL} = \Lambda_L \nabla u_{|L} \cdot n_{KL}$

$$u(y_\sigma) = \frac{\lambda_L d_{K,\sigma} u(x_L) + \lambda_K d_{L,\sigma} u(x_K)}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}}$$

Remark: if $\lambda_K^\sigma = \lambda_L^\sigma = 0$ then $\lambda_K \frac{y_K - y_\sigma}{d_{K,\sigma}} = \lambda_L \frac{y_\sigma - y_L}{d_{L,\sigma}}$ i.e. $\lambda_K \tan\theta_K = \lambda_L \tan\theta_L$

... no optics

Third step: complete definition of scheme



$$X_{\mathcal{B}} = \{(u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}, (u_{\sigma,s})_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}}$$
u_σ given by harmonic point averaging$\}$

$u_{K,s}^{\epsilon} = \dfrac{u_K + u_\tau}{2}$ if $\epsilon = [x_K, y_\tau]$ and
$u_{K,s}^{\epsilon} = u_{\tau,s}$ if $\epsilon = [s, y_\tau]$ for $\tau = \sigma$ and σ'

$$|K_s|\nabla_{K,s}u = \sum_{\epsilon\in\mathcal{E}_{K,s}} |\epsilon|(u_{K,s}^{\epsilon} - u_K)n_{K,s}^{\epsilon}$$

find $u \in X_{\mathcal{B}}$ s.t.
$\langle u, v\rangle_{\mathcal{D}} = \displaystyle\int_\Omega f(x)\Pi_{\mathcal{M}}v(x)\mathrm{d}x,\ \forall v \in X_{\mathcal{B}}$ with

$$\langle u, v\rangle_{\mathcal{D}} = \sum_{K\in\mathcal{M}}\sum_{s\in S_K}|K_s|\left(\Lambda_K\nabla_{K,s}u\cdot\nabla_{K,s}v + \sum_{\tau=\sigma,\sigma'}\alpha_{K\tau}R_{K,s}^{\tau}uR_{K,s}^{\tau}v\right)$$

with $\alpha_{K\tau} > 0$, $R_{K,s}^{\tau}u = \dfrac{1}{d_{K\tau}}(u_\tau - u_K - \nabla_{K,s}u\cdot(y_\tau - x_K))$, for $\tau = \sigma$ et σ'

properties of scheme: coercive, symmetric, convergent

Stencil of the scheme

scheme can be expressed by

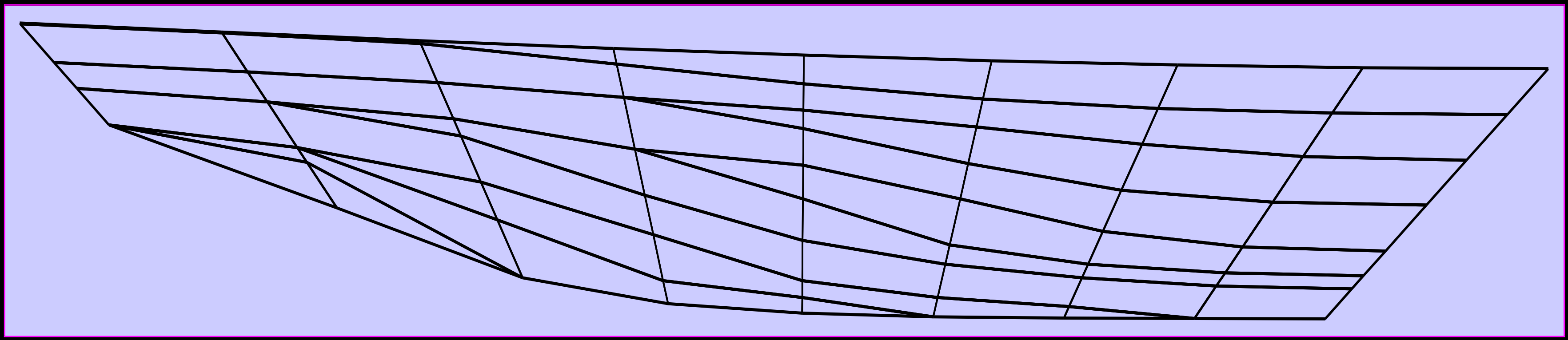
$$\forall K \in \mathcal{M}, \sum_{\substack{\sigma\in\mathcal{E}_K \\ \sigma=K|L}} F_{K,L}(u) + \sum_{\substack{\sigma\in\mathcal{E}_K \\ \sigma\subset\partial\Omega}} F_{K,\sigma}(u) = \int_K f(x)\mathrm{d}x$$

where expression of $F_{K,L}(u)$ can be deduced from

$$\langle u, v\rangle_{K,s} = \sum_{\epsilon\in\mathcal{E}_{K,s}} \sum_{\epsilon'\in\mathcal{E}_{K,s}} A_{K,s}^{\epsilon'\epsilon}(u_{K,s}^{\epsilon'} - u_K)(v_{K,s}^{\epsilon} - v_K) = \sum_{\epsilon\in\mathcal{E}_{K,s}} F_{K,s}^{\epsilon}(u)\ (v_{K,s}^{\epsilon} - v_K)$$

using conservation of fluxes and averaging expressions

Numerical results



$$u(x,y) = \sin(\pi x)\sin(\pi y)$$

	mesh 1	mesh 2	mesh 3	mesh 4	mesh 5
$\#\mathcal{M}$	62	302	1357	5363	21031
# hybrid edges	1	3	6	10	17
L^2-error	$9.15\ 10^{-3}$	$3.07\ 10^{-3}$	$9.30\ 10^{-4}$	$2.66\ 10^{-4}$	$6.89\ 10^{-5}$

Conclusions

DIOPTRE satisfies a few properties but no easy 3D extension

ideal scheme remains an open problem