Remarks on discretizations of convection-diffusion equations using hybrid mimetic mixed methods

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Joint works with:

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Contents

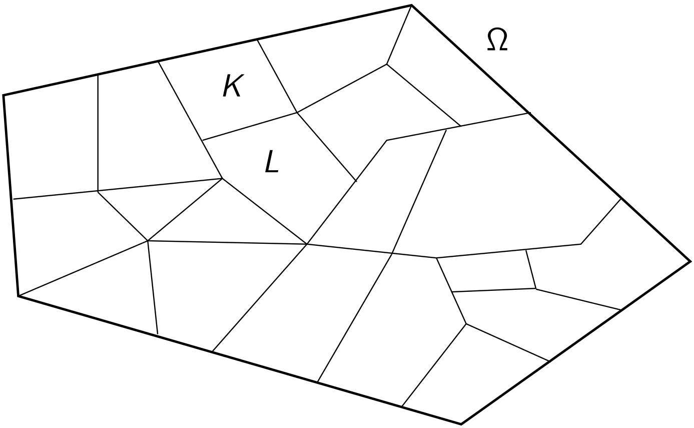
Equation

$$\begin{cases} -\mathrm{div}(\Lambda\nabla\bar{u}) + \mathrm{div}(V\bar{u}) = f & \text{in } \Omega, \\ \bar{u} = g & \text{on } \partial\Omega \end{cases}$$

with:

- Ω open subset of $\mathbb{R}^d$ $(d \geq 2)$,
- $\Lambda : \Omega \to M_d(\mathbb{R})$ a (symmetric) uniformly elliptic diffusion tensor,
- $V \in C^1(\overline{\Omega})^d$ with $\mathrm{div}(V) \geq 0$,
- $f \in L^2(\Omega)$,
- $g \in H^{1/2}(\partial\Omega)$ (often $g = 0$ for the theoretical study).

Grid



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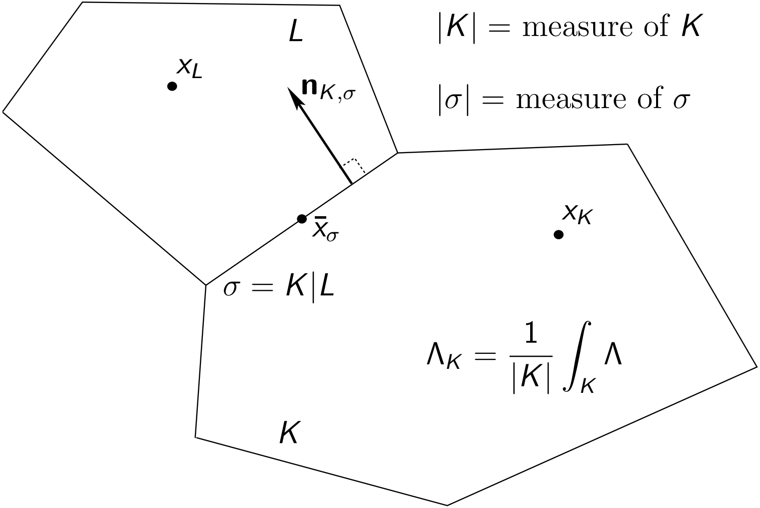
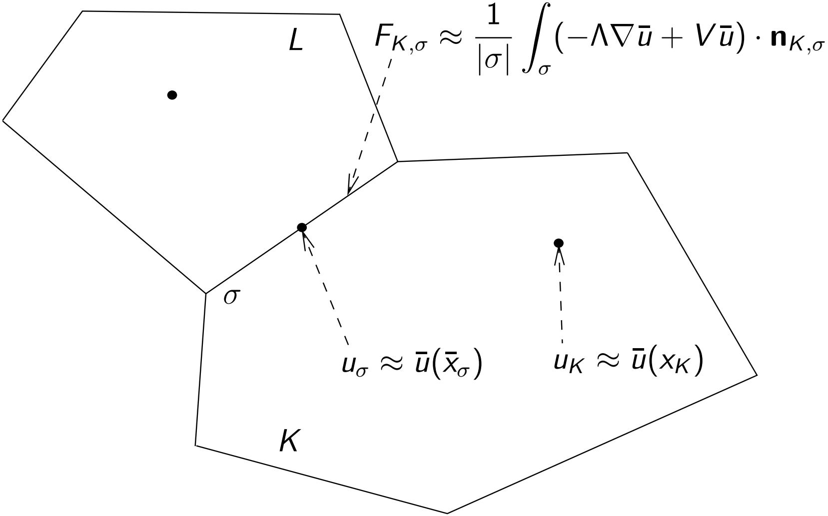


L
x_L
$\mathbf{n}_{K,\sigma}$
$|K|$ = measure of K
$|\sigma|$ = measure of σ
$\bar{x}_\sigma$
x_K
$\sigma = K|L$
$\Lambda_K = \frac{1}{|K|}\int_K \Lambda$
K

Unknowns



L
$F_{K,\sigma} \approx \frac{1}{|\sigma|} \int_\sigma (-\Lambda \nabla \bar{u} + V\bar{u}) \cdot \mathbf{n}_{K,\sigma}$
σ
$u_\sigma \approx \bar{u}(\bar{x}_\sigma)$
$u_K \approx \bar{u}(x_K)$
K

HMM schemes for purely diffusive equation (I)

We consider here:

$$\begin{cases} -\mathrm{div}(\Lambda \nabla \bar{u}) = f & \text{in } \Omega, \\ \bar{u} = g & \text{on } \partial\Omega \end{cases}$$

(i.e. $V = 0$)

HMM = Hybrid Mimetic Mixed: a single framework for three families of numerical methods:

- Hybrid Finite Volume
- Mimetic Finite Difference
- Mixed Finite Volume

HMM schemes (II): FV point of view

Flux balance

$$\forall K \ : \ \sum_{\sigma\in\mathcal{E}_K} |\sigma| F_{K,\sigma} = \int_K f$$

with $\mathcal{E}_K$= edges (faces in 3d) of the cell K.

Flux conservativity:

$$\forall\sigma \text{ between } K \text{ and } L \ : \ F_{K,\sigma} + F_{L,\sigma} = 0.$$

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Expression of the fluxes: for all K, let

$$\mathbf{v}_K(F_K) = \frac{1}{|K|}\Lambda_K^{-1} \sum_{\sigma\in\mathcal{E}_K} |\sigma| F_{K,\sigma}(x_K - \bar{x}_\sigma) \qquad (\approx (\nabla \bar{u})_{|K}),$$

$$\mathbb{T}_K F_K = \left(F_{K,\sigma} + \Lambda_K \mathbf{v}_K(F_K)\cdot \mathbf{n}_{K,\sigma}\right)_{\sigma\in\mathcal{E}_K},$$

HMM schemes (II): FV point of view

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$$\mathbb{T}_K F_K = (F_{K,\sigma} + \Lambda_K \mathbf{v}_K(F_K)\cdot\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K},$$

choose $\mathbb{B}_K$ s.p.d. and define $F_K = F_K(u_K,(u_\sigma)_{\sigma\in\mathcal{E}_K})$ by

$$\forall\sigma\in\mathcal{E}_K \; : \; u_K - u_\sigma = \mathbf{v}_K(F_K)\cdot(x_K - \bar{x}_\sigma) + \frac{1}{|\sigma|}(\mathbb{T}_K^T \mathbb{B}_K \mathbb{T}_K F_K)_\sigma.$$

HMM schemes (III): MFD point of view

Define:

- $Q_h =$ set of piecewise-constant functions on the grid,
- $X_h =$ set of *conservative* fluxes $(F_{K,\sigma})_{K,\sigma}$

Discrete divergence operator: let $\mathrm{div}_h : X_h \to Q_h$ be:

$$(\mathrm{div}_h(F))_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}.$$

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Scheme: find $u_h \in Q_h$ and $F_h \in X_h$ such that

$$\begin{aligned} \forall G \in X_h \quad &: \quad [F_h, G]_{X_h} = [\mathrm{div}_h(G), u_h]_{L^2}\,, \\ \forall q \in Q_h \quad &: \quad [\mathrm{div}_h(F_h), q]_{L^2} = [f_h, q]_{L^2} \end{aligned}$$

with $[\cdot,\cdot]_{X_h} = \sum_K [\cdot,\cdot]_K$ and

$$[F_K, G_K]_K = |K|\Lambda_K \mathbf{v}_K(F_K)\cdot \mathbf{v}_K(G_K) + (\mathbb{T}_K G_K)^T \mathbb{B}_K \mathbb{T}_K F_K$$

Some references on HMM (=Hybrid Mimetic Mixed)

- HFV:

Eymard, Gallouët, Herbin (2007) [*diffusion equations*]
Eymard, Gallouët, Herbin (2009) [*cell-centered alternative*]
Chénier, Eymard, Herbin (to appear) [*Navier-Stokes, cell-centered*]

- MFD:

Brezzi, Lipnikov, Shashkov (2005) [*Error estimates for diffusion*]
Brezzi, Lipnikov, Simoncini (2005) [*Family of methods*]
Beirão da Veiga, Manzini (2008) [*A posteriori estimators*]
Cangiani, Manzini (2008) [*Flux reconstruction and post-processing*]
Beirão da Veiga (to appear) [*Linear elasticity*]

- MFV:

D., Eymard [*diffusion equations*]
D. (2006) [*fully non-linear p-Laplacian type equations*]
Chainais-Hillairet, D. (2007) [*miscible flow in porous medium*]
D., Eymard (2009) [*Navier-Stokes equation*]

- FVCA5 benchmark: Chainais-Hillairet, D., Eymard, Gallouët, Herbin, Lipnikov, Manzini.

Contents

Convection-diffusion equation

We come back to :

$$\begin{cases} -\text{div}(\Lambda \nabla \bar{u}) + \text{div}(V \bar{u}) = f & \text{in } \Omega, \\ \bar{u} = g & \text{on } \partial\Omega \end{cases}$$

Contents

Principle of FV-based discretizations

- Keep the same expression for the *diffusive* flux $F_{K,\sigma}$:

$$\forall \sigma \in \mathcal{E}_K \;:\; u_K - u_\sigma = \mathbf{v}_K(F_K) \cdot (x_K - \bar{x}_\sigma) + \frac{1}{|\sigma|}(\mathbb{T}_K^T \mathbb{B}_K \mathbb{T}_K F_K)_\sigma.$$

- Using $(u_K)_K$ and $(u_\sigma)_\sigma$, compute some discretizations of the convective fluxes

$$F^c_{K,\sigma} \approx \frac{1}{|\sigma|} \int_\sigma \bar{u} V_{K,\sigma} \cdot \mathbf{n}_{K,\sigma}.$$

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- Keep the same expression for the *diffusive* flux $F_{K,\sigma}$:

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- Using $(u_K)_K$ and $(u_\sigma)_\sigma$, compute some discretizations of the convective fluxes

$$F_{K,\sigma}^c \approx \frac{1}{|\sigma|}\int_\sigma \bar{u} V_{K,\sigma}\cdot \mathbf{n}_{K,\sigma}.$$

- **HMM scheme with FV handling of the convection**:

$$\forall K \;:\; \sum_{\sigma\in\mathcal{E}_K} |\sigma|(F_{K,\sigma} + F_{K,\sigma}^c) = \int_K f,$$

$$\forall \sigma \text{ between } K \text{ and L} \;:\; (F_{K,\sigma} + F_{K,\sigma}^c) + (F_{L,\sigma} + F_{L,\sigma}^c) = 0.$$

Classical FV methods for convection

If $V_{K,\sigma} = \frac{1}{|\sigma|}\int_\sigma V \cdot \mathbf{n}_{K,\sigma}$ and σ is an edge between K and L,

- Centered scheme:

$$F^c_{K,\sigma} = V_{K,\sigma}\frac{u_K + u_L}{2}.$$

- Upwind scheme:

$$F^c_{K,\sigma} = \begin{cases} V_{K,\sigma} u_K & \text{if } V_{K,\sigma} \geq 0\,, \\ V_{K,\sigma} u_L & \text{if } V_{K,\sigma} < 0. \end{cases}$$

- Scharfetter-Gummel scheme:

$$F^c_{K,\sigma} = \frac{1}{d_\sigma}\left(A_{\mathrm{sg}}(d_\sigma V_{K,\sigma})u_K - A_{\mathrm{sg}}(-d_\sigma V_{K,\sigma})u_L\right)$$

with $d_\sigma = d(x_K, x_L)$ and $A_{\mathrm{sg}}(s) = \frac{-s}{e^{-s}-1} - 1$.

Remark: scaling of the Scharfetter-Gummel method

In case of strong convection: stabilization of the method by taking into account the ratio diffusion/convection.

$$A_{\mathrm{sg}}(s) \rightsquigarrow A_{\mathrm{sg},\lambda_\sigma}(s) = \lambda_\sigma A_{\mathrm{sg}}\left(\frac{s}{\lambda_\sigma}\right)$$

with λ_σ is the smallest eigenvalue of Λ_K and Λ_L.

Alternative: edge-based convective fluxes

HMM methods are *hybrid* methods, with edge unknowns.

Alternative: edge-based convective fluxes

HMM methods are *hybrid* methods, with edge unknowns.

Use of the edge unknowns: instead of u_L, take u_σ in the numerical convective fluxes.

Example: for the upwind scheme,

$$F^c_{K,\sigma} = \begin{cases} V_{K,\sigma} u_K & \text{if } V_{K,\sigma} \geq 0\,, \\ V_{K,\sigma} u_\sigma & \text{if } V_{K,\sigma} < 0. \end{cases}$$

- Convective fluxes no longer conservative ($F^c_{K,\sigma} + F^c_{L,\sigma} \neq 0$ in general).
- But the resulting scheme can be hybridized: local eliminations lead to a system in the edge unknowns $(u_\sigma)_\sigma$ only.

Contents

MFD formulation of the HMM method, purely diffusive problems

Original equation

$$-\mathrm{div}(\Lambda\nabla\bar{u}) = f.$$

Mixed formulation: with $\overline{F} = -\Lambda\nabla\bar{u}$,

$$\begin{aligned} \forall \mathbf{w} \in H(\mathrm{div},\Omega) &\ :\ [\Lambda^{-1}\overline{F}, \mathbf{w}]_{L^2} = [\mathrm{div}(\mathbf{w}), \bar{u}]_{L^2}\,, \\ \forall q \in L^2(\Omega) &\ :\ [\mathrm{div}(\overline{F}), q]_{L^2} = [f, q]_{L^2} \end{aligned}$$

MFD formulation of the HMM method

$$\begin{aligned} \forall G \in X_h &\ :\ [F_h, G]_{X_h} = [\mathrm{div}_h(G), u_h]_{L^2}\,, \\ \forall q \in Q_h &\ :\ [\mathrm{div}_h(F_h), q]_{L^2} = [f_h, q]_{L^2} \end{aligned}$$

MFD formulation of the HMM method, convection-diffusion problems

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$$\begin{aligned}
\forall \mathbf{w} \in H(\mathrm{div},\Omega) \quad &: \quad [\Lambda^{-1}\overline{F}, \mathbf{w}]_{L^2} = [\mathrm{div}(\mathbf{w}), \bar{u}]_{L^2} + [\Lambda^{-1}V\bar{u}, \mathbf{w}]_{L^2}\,, \\
\forall q \in L^2(\Omega) \quad &: \quad [\mathrm{div}(\overline{F}), q]_{L^2} = [f, q]_{L^2}
\end{aligned}$$

MFD formulation of the HMM method

$$\begin{aligned}
\forall G \in X_h \quad &: \quad [F_h, G]_{X_h} = [\mathrm{div}_h(G), u_h]_{L^2} + ???\,, \\
\forall q \in Q_h \quad &: \quad [\mathrm{div}_h(F_h), q]_{L^2} = [f_h, q]_{L^2}
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\end{aligned}$$

Discretization of the convection term

$$[\Lambda^{-1} V \bar{u}, \mathbf{w}]_{L^2} = \sum_K \int_K \Lambda^{-1} V \bar{u} \cdot \mathbf{w} \approx \sum_K u_K \int_K \Lambda^{-1} V \cdot \mathbf{w}.$$

And from $[\Lambda^{-1} \overline{F}, \mathbf{w}]_{L^2} \rightsquigarrow [F_h, G]_{X_h}$,

$$\int_K \Lambda^{-1} V \cdot \mathbf{w} \approx [V^I, G]_K.$$

The resulting scheme

- **HMM scheme with MFD handling of the convection**: find $u_h \in Q_h$ and $F_h \in X_h$ such that

$$\begin{aligned} \forall G \in X_h \quad &: \quad [F_h, G]_{X_h} = [\mathrm{div}_h(G), u_h]_{L^2} + \sum_K u_K [V^I, G]_K \,, \\ \forall q \in Q_h \quad &: \quad [\mathrm{div}_h(F_h), q]_{L^2} = [f_h, q]_{L^2}. \end{aligned}$$

Ref.: Cangiani-Manzini-Russo, 2009 (to appear)

Stabilization for convection-dominated situations

Take $\alpha > 0$ and define $J_h : Q_h \to Q_h$ by

$$J_h(q)_K = \frac{\alpha}{2|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma|\,|V_{K,\sigma}|(q_K - q_L).$$

HMM scheme, with stabilized MFD handling of the convection: find $u_h \in Q_h$ and $F_h \in X_h$ such that

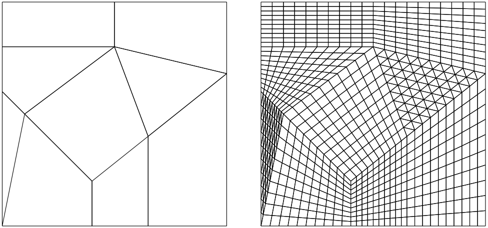
$$\begin{aligned} \forall G \in X_h \quad &: \quad [F_h, G]_{X_h} = [\mathrm{div}_h(G), u_h]_{L^2} + \sum_K u_K [V^I, G]_K\,, \\ \forall q \in Q_h \quad &: \quad [\mathrm{div}_h(F_h) + J_h(u_h), q]_{L^2} = [f_h, q]_{L^2}. \end{aligned}$$

Contents

Orders of convergence

Data (heterogeneous anisotropic but regular)

- $\Omega = (0,1)^2$,
- $\Lambda(x,y) =$ matrix $\nu \times \mathrm{diag}(2,1)$ rotated of an angle $2\pi x$,
- $V(x,y) = 10(-y,x)$,
- $\bar{u}(x,y) = x(1-x)e^y$.
- Grid ($n = 25, 50, 100, 150$):

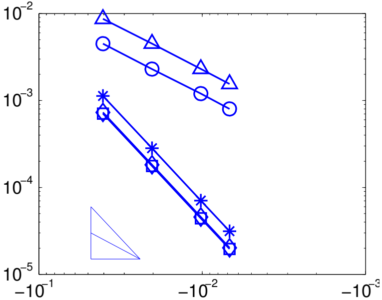


$n = 1$ $\qquad$ $n = 10$

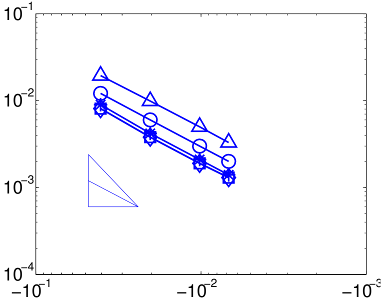
If $\nu \ll 1$: convection-dominated.

Orders of convergence

Diffusive regime: $\nu = 1$ (reference slopes: h and h^2)



L^2 error on u

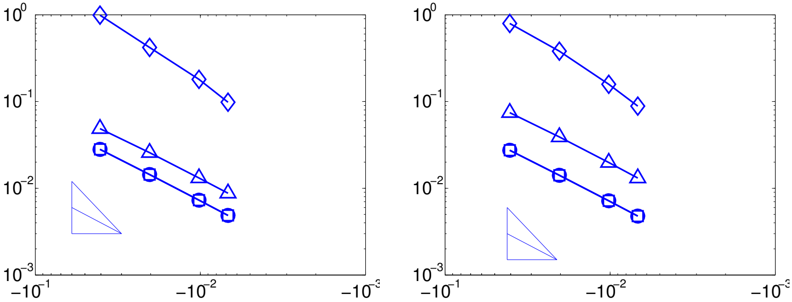


L^2 error on ∇u

$\bigcirc$ FV-upwind, $\square$ FV-Scharfetter-Gummel, $\diamond$ FV-centered,
$\star$ MFD-nonstabilized, $\triangle$ MFD-stabilized.

Orders of convergence

Convective regime: $\nu = 10^{-4}$ (reference slopes: h and h^2)



L^2 error on u $\qquad$ X_h error on the (full) fluxes

◯ FV-upwind, □ FV-Scharfetter-Gummel (scaled),
◇ FV-centered, △ MFD-stabilized.

FV methods: about the choice of the upwinding

Cell-upwind

$$F^c_{K,\sigma} = \begin{cases} V_{K,\sigma} u_K & \text{if } V_{K,\sigma} \geq 0\,, \\ V_{K,\sigma} u_L & \text{if } V_{K,\sigma} < 0. \end{cases}$$

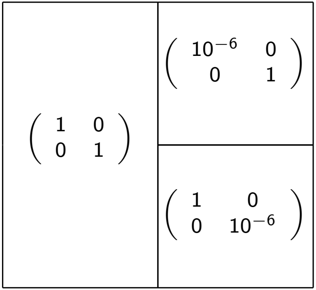
Edge-upwind

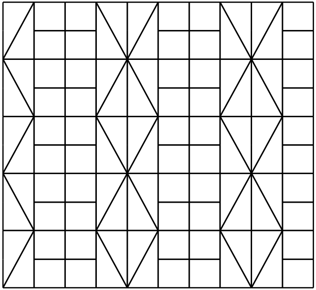
$$F^c_{K,\sigma} = \begin{cases} V_{K,\sigma} u_K & \text{if } V_{K,\sigma} \geq 0\,, \\ V_{K,\sigma} u_\sigma & \text{if } V_{K,\sigma} < 0. \end{cases}$$

FV methods: about the choice of the upwinding

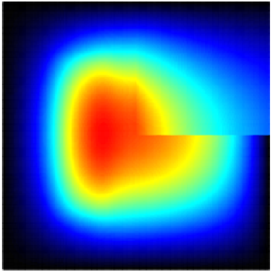
Data

- $V(x, y) = (5, 0)$,
- Source $f = 1$ on $(0.2; 0.4) \times (0.2; 0.8)$, $f = 0$ elsewhere; homogeneous boundary conditions.
- Diffusion and grid pattern:

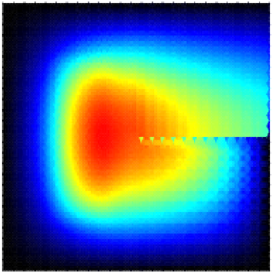




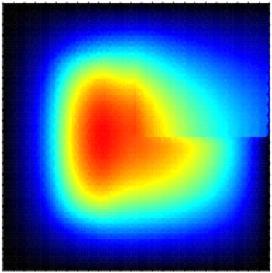
FV methods: about the choice of the upwinding



Exact solution



Cell upwind



Edge upwind

Remarks on the scaling of the SG function (I)

Scaled Scharfetter-Gummel method:

$$F^c_{K,\sigma} = \frac{1}{d_\sigma}\left(A_{\mathrm{sg},\lambda_\sigma}(d_\sigma V_{K,\sigma})u_K - A_{\mathrm{sg},\lambda_\sigma}(-d_\sigma V_{K,\sigma})u_L\right)$$

with λ_σ the local minimum of the eigenvalues of Λ.

Asymptotic behavior: if $\lambda_\sigma \ll 1$,

$$F^c_{K,\sigma} \approx \begin{cases} V_{K,\sigma}u_K & \text{if } V_{K,\sigma} \geq 0\,, \\ V_{K,\sigma}u_L & \text{if } V_{K,\sigma} < 0. \end{cases}$$

- This is the flux of the upwind scheme.

Remarks on the scaling of the SG function (II)

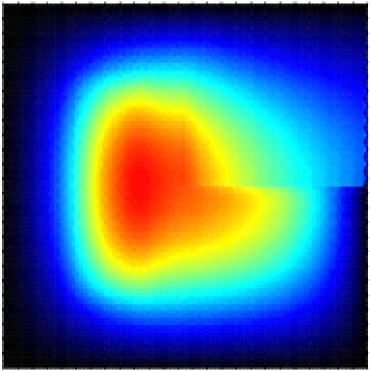
Same test as before, two different scalings for SG.

$$\lambda_\sigma = \min \mathrm{Sp}(\Lambda) \qquad\qquad \lambda_\sigma = \Lambda \mathbf{n}_\sigma \cdot \mathbf{n}_\sigma$$

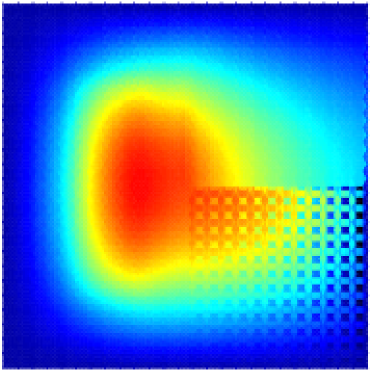
Remarks on the scaling of the SG function (II)

Same test as before, two different scalings for SG.

$\lambda_\sigma = \min \mathrm{Sp}(\Lambda)$



$\lambda_\sigma = \Lambda \mathbf{n}_\sigma \cdot \mathbf{n}_\sigma$



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Unified presentation of the discretization

All the (cell-based) numerical convective flux can be written

$$F^c_{K,\sigma} = \frac{1}{d_\sigma}\left(A(d_\sigma V_{K,\sigma})u_K + B(d_\sigma V_{K,\sigma})u_L\right)$$

Centered:	$A(s) = \frac{s}{2}$	$B(s) = \frac{s}{2}$
Upwind:	$A(s) = s^+$	$B(s) = s^-$
Scharfetter-Gummel:	$A = A_{\mathrm{sg}}$	$B(s) = -A_{\mathrm{sg}}(-s)$
Stabilized MFD:	$A(s) = s + \frac{\alpha}{2}\lvert s\rvert$	$B(s) = -\frac{\alpha}{2}\lvert s\rvert$

Generic convergence and error estimates

Stability and existence of a solution: for all h (for FV-like methods) or h **small enough** (for the stabilized MFD method),

$$||u_h||_{L^2(\Omega)} + ||F_h||_{X_h} \leq C||f||_{L^2(\Omega)}.$$

Convergence without regularity: for $\bar{u} \in H_0^1(\Omega)$, as $h \to 0$,

$$\begin{aligned} u_h &\to \bar{u} \qquad \text{in } L^r(\Omega) \text{ for } r < \tfrac{2d}{d-2}\,, \\ \mathbf{v}(F_h) &\to \nabla \bar{u} \qquad \text{in } (L^2(\Omega))^d. \end{aligned}$$

Error estimates: if $\bar{u} \in H^2(\Omega)$, for $r < \frac{2d}{d-2}$:

$$||u_h - \bar{u}||_{L^r(\Omega)} + ||F_h - \bar{F}||_{X_h} = \mathcal{O}(h).$$

Thanks.