Finite element methods for the Brinkman model

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The Brinkman model

- Describes the flow of a viscous fluid in a porous medium
- Applicable to materials of very high porosity, e.g.
  - Sands, porous stones, petroleum engineering
  - Heat pipes
- Often used as a transition layer between free and porous flow
- Obtained from Navier-Stokes by homogenization.
  Allaire 1990, Arbogast, Lehr 2006, ...

The scaled Brinkman problem

- Darcy: $$\mathbf{u} + \nabla p = \mathbf{0} \text{ in } \Omega$$ $$\operatorname{div} \mathbf{u} = g \text{ in } \Omega$$
- Stokes: $$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega$$ $$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega$$
- Brinkman: $$-t^2 \Delta \mathbf{u} + \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega$$ $$\operatorname{div} \mathbf{u} = g \text{ in } \Omega$$
- $0 \leq t \leq C$: $* \, t = 0 \sim$ the Darcy limit
  $* \, t$ small $\sim$ Singularly perturbed Darcy
  $* \, t >> 0 \sim$ Stokes type problem

The weak form of the Brinkman problem

- Assume homogenous boundary conditions and $p \in L_0^2(\Omega)$.

- Define
$$a(\mathbf{u}, \mathbf{v}) = t^2 (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbf{u}, \mathbf{v})$$
$$b(p, \mathbf{v}) = -(\mathrm{div}\, \mathbf{v}, p)$$

- and
$$\mathcal{B}(\mathbf{u}, p; \mathbf{v}, q) = a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u})$$
$$\mathcal{L}(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) - (g, q)$$

- Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that
$$\mathcal{B}(\mathbf{u}, p; \mathbf{v}, q) = \mathcal{L}(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q$$

The norms and spaces

- $$a(\mathbf{u}, \mathbf{v}) = t^2 (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbf{u}, \mathbf{v})$$
  $$\|\mathbf{v}\|_t^2 = t^2 \|\nabla \mathbf{v}\|_0^2 + \|\mathbf{v}\|_0^2$$
- $$b(p, \mathbf{v}) = -(\operatorname{div} \mathbf{v}, p)$$
  $$|||q|||_t = \sup_{\mathbf{v} \in \mathbf{V}} \frac{(\operatorname{div} \mathbf{v}, q)}{\|v\|_t}$$
- $\mathbf{V}$ is the completion of $[C_0^\infty(\Omega)]^N$ with respect to $\| \cdot \|_t$
  $Q = \{ \, q \in L_0^2(\Omega) \mid \; |||q|||_t < \infty \, \}$.

The norms and spaces

- $$\|\mathbf{v}\|_t^2 = t^2\|\nabla \mathbf{v}\|_0^2 + \|\mathbf{v}\|_0^2 \qquad |||q|||_t = \sup_{\mathbf{v}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{v}, q)}{\|v\|_t}$$
- For $t > 0$ (i.e. Stokes): $\mathbf{V} = [H_0^1(\Omega)]^N \quad Q = L_0^2(\Omega)$
  For $t = 0$ (i.e. Darcy): $\mathbf{V} = [L^2(\Omega)]^N \quad Q = H^1(\Omega) \cap L_0^2(\Omega)$
- But not uniformly !!
  $C_1 t\|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_t \leq C_2\|\mathbf{v}\|_1$
  $C_1\|q\|_0 \leq |||q|||_t \leq C_2 t^{-1}\|q\|_0$
- Boundary conditions in **V**: $*$ $t > 0$: homogenous Dirichlet
  $*$ $t = 0$: natural Neumann

The Babuska-Brezzi conditions

- By definition it holds

$$a(\mathbf{v},\mathbf{v}) \geq \|\mathbf{v}\|_t^2 \quad \forall \mathbf{v} \in \mathbf{V}$$

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{v}, q)}{\|\mathbf{v}\|_t} \geq |||q|||_t \quad \forall q \in Q$$

- Hence

$$\sup_{(\mathbf{v},q)\in\mathbf{V}\times Q} \frac{\mathcal{B}(\mathbf{w},r;\mathbf{v},q)}{\|\mathbf{v}\|_t + |||q|||_t} \geq C\left(\|\mathbf{w}\|_t + |||r|||_t\right) \quad \forall (\mathbf{w},r) \in \mathbf{V}\times Q$$

- We have a unique solution for all $0 \leq t \leq C$!!

Mesh dependent pressure norm

- The pressure norm: $|||q|||_t = \sup_{\mathbf{v}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{v}, q)}{\|v\|_t}$
- The discrete counterpart: $|||q|||_{t,h}^2 = \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} \|\nabla q\|_{0,K}^2$
- $t = 0 : \Rightarrow \quad |||q|||_t = \|q\|_1 = |||q|||_{t,h}$
- $t \gg 0 : \Rightarrow \quad |||q|||_t = \|q\|_0 = \|\nabla q\|_{-1}$

  $|||q|||_{t,h} = h\|\nabla q\|_0 = \|\nabla q\|_{-1,h}$
- Why do we use this norm?
  – Needed in the stability analysis of the discretization
  – Basis for a posteriori estimates
  – Computable

How to discretize?

- Stokes elements (e.g. MINI, Taylor–Hood)
  $+$ check stability in the Darcy limit
- Stabilize

Discretizations with Stokes elements

- MINI:

$$\mathbf{V}_h = \{\mathbf{v} \in [C(\Omega)]^N \cap \mathbf{V} \mid \mathbf{v}|_K \in [P_k(K) \cup B_{k+N}(K)]^N\},$$
$$Q_h = \{q \in C(\Omega) \cap L_0^2(\Omega) \mid q|_K \in P_k(K)\},$$

- Taylor-Hood:

$$\mathbf{V}_h = \{\mathbf{v} \in [C(\Omega)]^N \cap \mathbf{V} \mid \mathbf{v}|_K \in [P_{k+1}(K)]^N\},$$
$$Q_h = \{q \in C(\Omega) \cap L_0^2(\Omega) \mid q|_K \in P_k(K)\}.$$

- The Brezzi condition OK in $\mathbf{V}_h \times Q_h$
  –in the norm $\|\mathbf{v}\|_t + |||q|||_t$
  –in the norm $\|\mathbf{v}\|_t + |||q|||_{t,h}$
  –for all $0 \leq t \leq C$

The stabilized discretizations

Stabilized method: Find $(\mathbf{u}_h, p_h)$ such that

$$\begin{aligned}
&\mathcal{B}(\mathbf{u}_h, p_h; \mathbf{v}, q) \\
&\quad - \alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} \left(t^2 \Delta \mathbf{u}_h - \mathbf{u}_h - \nabla p_h, t^2 \Delta \mathbf{v} - \mathbf{v} - \nabla q \right)_K \\
&= \mathcal{L}(\mathbf{v}, q) - \alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} \left(\mathbf{f}, t^2 \Delta \mathbf{v} - \mathbf{v} - \nabla q \right)_K
\end{aligned}$$

Spaces:

$$\mathbf{V}_h = \{ \mathbf{v} \in [C(\Omega)]^N \cap \mathbf{V} \mid \mathbf{v}|_K \in [P_k(K)]^N \}$$

$$Q_h = \{ q \in C(\Omega) \cap L_0^2(\Omega) \mid q|_K \in P_k(K) \}$$

Stability OK in the norm $\|\mathbf{v}\|_t + |||q|||_{t,h}$

A priori results for the smooth solution

- MINI and stabilized methods:

$$\|\mathbf{u} - \mathbf{u}_h\|_t + |||p - p_h|||_{t,h} \leq C\big((t+h)h^k\|\mathbf{u}\|_{k+1} + (t+h)^{-1}h^{k+1}\|p\|_{k+1}\big).$$

$\Rightarrow \mathcal{O}(h^k)$ independent of t.

- Taylor-Hood:

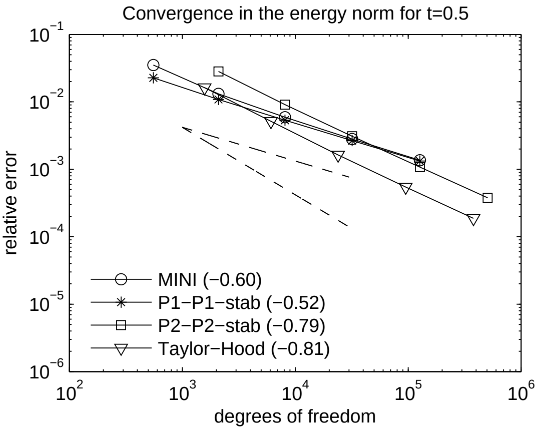
$$\|\mathbf{u} - \mathbf{u}_h\|_t + |||p - p_h|||_{t,h} \leq C\big((t+h)h^{k+1}\|\mathbf{u}\|_{k+2} + (t+h)^{-1}h^{k+1}\|p\|_{k+1}\big).$$

$\Rightarrow \mathcal{O}(h^{k+1})$ if $t > h$,
$\mathcal{O}(h^k)$ otherwise (i.e. in or near the Darcy limit)

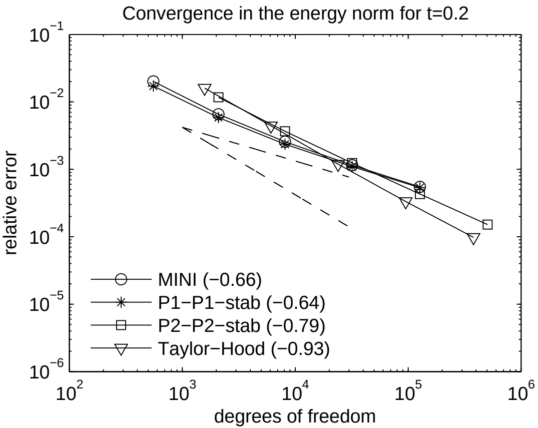
Numerical example of a smooth solution

- Unit square
- $$p(x,y) = -\sin(x)\sinh(y) + C$$
- $$\mathbf{u} = -\nabla p(x,y) = \begin{pmatrix} \cos(x)\sinh(y) \\ \sin(x)\cosh(y) \end{pmatrix}$$

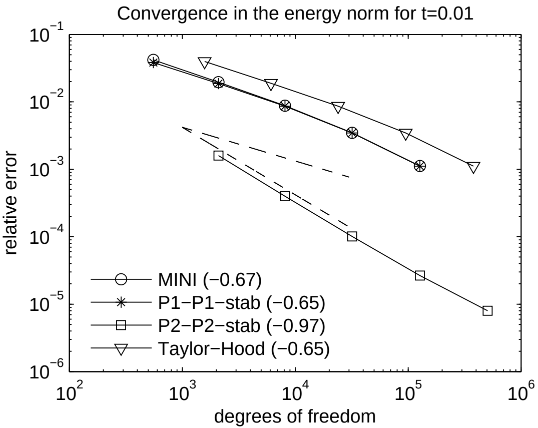
Convergence for the smooth solution



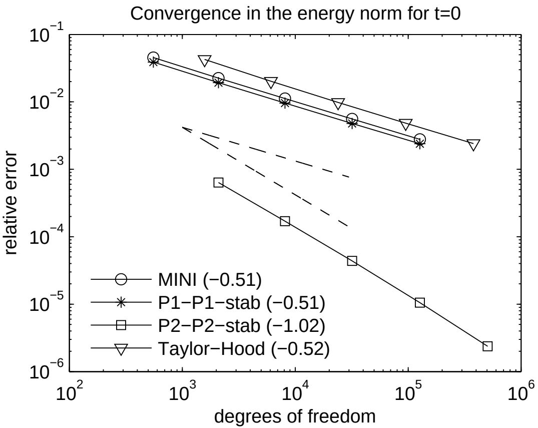
Convergence for the smooth solution



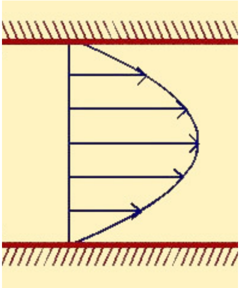
Convergence for the smooth solution

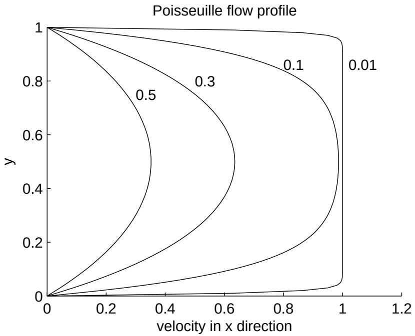


Convergence for the smooth solution



The Poiseuille flow



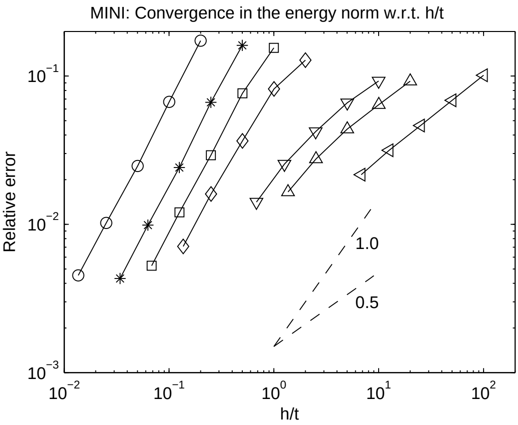


Problem: $-t^2 u''(y) + u(y) = 1 \quad$ with $\quad u(0) = u(1) = 0$

Solution:

$$u(y) = \begin{cases} \left(1 + e^{1/t} - e^{(1-y)/t} - e^{y/t}\right) / \left(1 + e^{1/t}\right) & \text{if } t > 0, \\ 1 \quad \text{if } t = 0. \end{cases}$$

Convergence for the Poiseuille flow



Stokes Darcy

A posteriori

$$\|\mathbf{u} - \mathbf{u}_h\|_t + |||p - p_h|||_{t,h} \leq C\Big(\sum_{K \in \mathcal{C}_h} E_K(\mathbf{u}_h, p_h)^2 \Big)^{1/2}$$

$$E_K(\mathbf{u}_h, p_h)^2 = \frac{h_K^2}{t^2 + h_K^2} \|t^2 \Delta \mathbf{u}_h - \mathbf{u}_h - \nabla p_h + \mathbf{f}\|_{0,K}^2$$
$$+ (t^2 + h_K^2) \|\mathrm{div}\, \mathbf{u}_h - g\|_{0,K}^2$$
$$+ \frac{h_K}{t^2 + h_K^2} \|[\![t^2 \partial_n(\mathbf{u}_h)]\!]\|_{0,\partial K \setminus \partial\Omega}^2 + \frac{t^2 + h_K^2}{h_K} \|(\mathbf{u}_h - \mathbf{u}_D) \cdot \mathbf{n}\|_{0,\partial K \cap \partial\Omega}^2$$

- Holds independent of t
- Also a lower bound
- Generalizes the Stokes and Darcy results

A posteriori in the Darcy limit

Original:

$$E_K(\mathbf{u}_h, p_h)^2 = \frac{h_K^2}{t^2 + h_K^2}\|t^2\Delta\mathbf{u}_h - \mathbf{u}_h - \nabla p_h + \mathbf{f}\|_{0,K}^2$$
$$+ (t^2 + h_K^2)\|\mathrm{div}\,\mathbf{u}_h - g\|_{0,K}^2$$
$$+ \frac{h_K}{t^2 + h_K^2}\|[\![t^2\partial_n(\mathbf{u}_h)]\!]\|_{0,\partial K\setminus\partial\Omega}^2 + \frac{t^2 + h_K^2}{h_K}\|(\mathbf{u}_h - \mathbf{u}_D)\cdot\mathbf{n}\|_{0,\partial K\cap\partial\Omega}^2$$

As $t \rightarrow 0$:

$$E_K(\mathbf{u}_h, p_h)^2 = \| - \mathbf{u}_h - \nabla p_h + \mathbf{f}\|_{0,K}^2 + h_K^2\|\mathrm{div}\,\mathbf{u}_h - g\|_{0,K}^2$$
$$+ h_K\|(\mathbf{u}_h - \mathbf{u}_D)\cdot\mathbf{n}\|_{0,\partial K\cap\partial\Omega}^2$$

A posteriori for $t \approx 1$

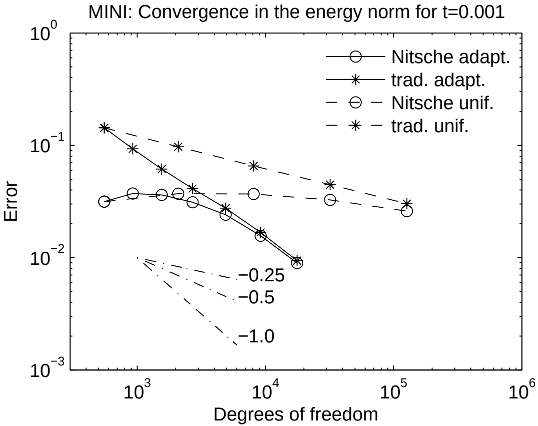
Original:

$$E_K(\mathbf{u}_h, p_h)^2 = \frac{h_K^2}{t^2 + h_K^2} \| t^2 \Delta \mathbf{u}_h - \mathbf{u}_h - \nabla p_h + \mathbf{f} \|_{0,K}^2$$
$$+ (t^2 + h_K^2) \| \mathrm{div}\, \mathbf{u}_h - g \|_{0,K}^2$$
$$+ \frac{h_K}{t^2 + h_K^2} \| [\![t^2 \partial_n (\mathbf{u}_h)]\!] \|_{0,\partial K \setminus \partial \Omega}^2 + \frac{t^2 + h_K^2}{h_K} \| (\mathbf{u}_h - \mathbf{u}_D) \cdot \mathbf{n} \|_{0,\partial K \cap \partial \Omega}^2$$

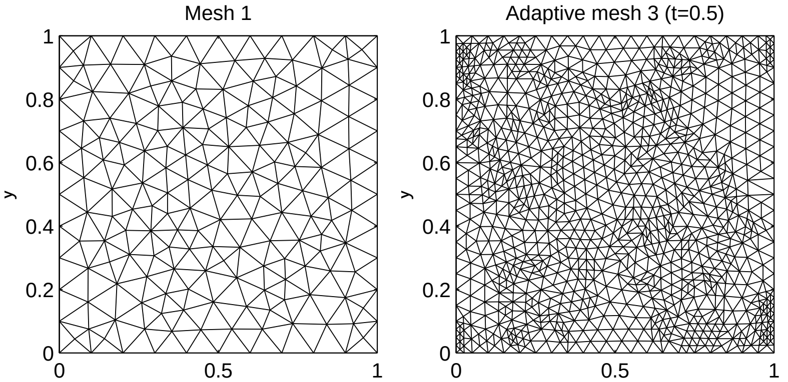
As $t \approx 1$:

$$E_K(\mathbf{u}_h, p_h)^2 \approx h_K^2 \| t^2 \Delta \mathbf{u}_h - \mathbf{u}_h - \nabla p_h + \mathbf{f} \|_{0,K}^2$$
$$+ \| \mathrm{div}\, \mathbf{u}_h - g \|_{0,K}^2$$
$$+ h_K \| [\![t^2 \partial_n (\mathbf{u}_h)]\!] \|_{0,\partial K \setminus \partial \Omega}^2 + \frac{t^2}{h_K} \| (\mathbf{u}_h - \mathbf{u}_D) \cdot \mathbf{n} \|_{0,\partial K \cap \partial \Omega}^2$$

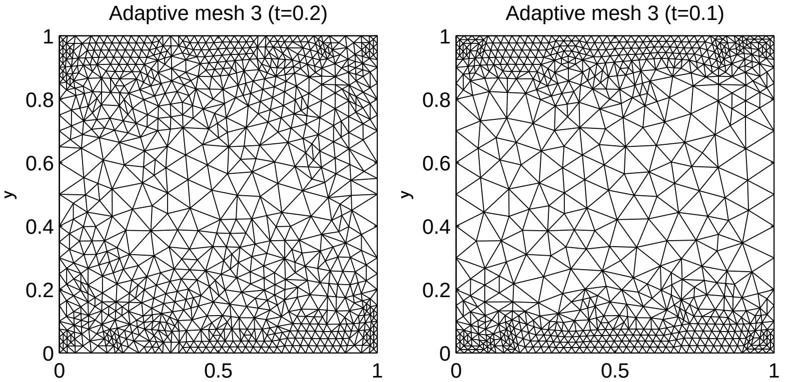
Adaptive refinement for the Poiseuille flow



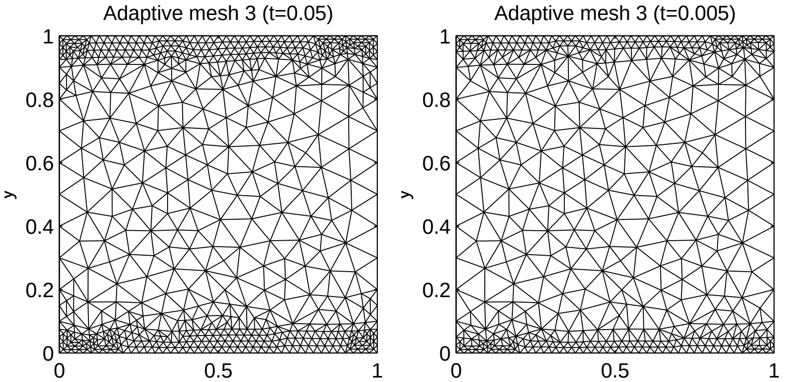
Adaptive refinement for the Poiseuille flow



Adaptive refinement for the Poiseuille flow



Adaptive refinement for the Poiseuille flow



A third family. Raviart-Thomas type elements

- Mixed methods of Raviart-Thomas type are standard for the Darcy equations
- Nonconforming for Brinkman and Stokes
- Remedy: Discontinuous Galerkin (Nietsche).
- Postprocessing of the pressure is the key to a posteriori estimates (Lovadina, Stenberg 2006)

The Mixed FE Spaces

- We use the BDM/BDDF spaces of order k

$$\mathbf{V}_h = \{\mathbf{v} \in H(\mathrm{div}\,, \Omega) \mid \mathbf{v}|_K \in [P_k(K)]^n \ \forall K \in \mathcal{C}_h\},$$
$$Q_h = \{q \in L^2(\Omega) \mid q|_K \in P_{k-1}(K) \ \forall K \in \mathcal{C}_h\}.$$

- This pairing satisfies the equilibrium property $\mathrm{div}\, \mathbf{V}_h \subset Q_h$
- Only the normal component of the flux is continuous

The DG formulation

- The bilinear form a is replaced with

$$a_h(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{v}) + t^2\Big(\sum_{K\in\mathcal{C}_h} (\nabla\mathbf{u},\nabla\mathbf{v})_K + \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_E}(\langle[\![\mathbf{u}]\!],[\![\mathbf{v}]\!]\rangle_E \\ - \langle\{\frac{\partial\mathbf{u}}{\partial n}\},[\![\mathbf{v}]\!]\rangle_E - \langle[\![\mathbf{u}]\!],\{\frac{\partial\mathbf{v}}{\partial n}\}\rangle_E)\Big),$$

  with $\alpha > 0$ big enough. Here $[\![\cdot]\!]$ = jump and $\{\cdot\}$ = average.
- This, however, not enough!

Postprocessing of the pressure

- The "raw" pressure is in $(k \geq 1)$

$$Q_h = \{q \in L^2(\Omega) \mid q|_K \in P_{k-1}(K)\ \forall K \in \mathcal{C}_h\}.$$

- The postprocessed pressure is sought from

$$Q_h^* = \{q \in L^2(\Omega) \mid q|_K \in P_{k+1}(K)\ \forall K \in \mathcal{C}_h\}.$$

- Let $P_h : L^2(\Omega) \to Q_h$ be the L^2-projection.

The exact pressure satisfies

$$\nabla p = \mathbf{f} + t^2 \Delta \mathbf{u} - \mathbf{u}.$$

The postprocessing is: find $p_h^* \in Q_h^*$ such that

$$\begin{aligned} P_h p_h^* &= p_h, \\ (\nabla p_h^*, \nabla q)_K &= (\mathbf{f} + t^2 \Delta \mathbf{u}_h - \mathbf{u}_h, \nabla q)_K \quad \forall q \in (I - P_h) Q_h^*|_K \end{aligned}$$

for all $K \in \mathcal{C}_h$.

Note that this is done element by element.

: The jumps in the (tangential component) of the velocity and the jumps in the pressure have to be added:

$$\|\mathbf{v}\|_{t,h}^2 = \|\mathbf{v}\|_0^2 + t^2\Big(\sum_{K\in\mathcal{C}_h} \|\nabla \mathbf{v}\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|[\![\mathbf{v}]\!]\|_{0,E}^2 \Big)$$

$$|||q|||_{t,h}^2 = \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|\nabla q\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} \frac{h_E}{t^2+h_E^2} \|[\![q]\!]\|_{0,E}^2$$

The coercivty is DG standard. The Babuska-Brezzi follows from the definition of the DOF's of BDM.

A priori estimates

From the stability, consistency and the equilibrium property it follows

$$\|\mathbf{u} - \mathbf{u}_h\|_{t,h} + |||P_h p - p_h|||_{t,h} \le C\|\mathbf{u} - \mathbf{R}_h\mathbf{u}\|_{t,h},$$

where $\mathbf{R}_h$ is the BDM interpolation operator. For a smooth solution this gives

$$\|\mathbf{u} - \mathbf{u}_h\|_{t,h} + |||P_h p - p_h|||_{t,h} \le C(t+h)h^k\|\mathbf{u}\|_{k+1}.$$

Note: 1. In the Darcy limit this improves the MINI/Stabilzed.
2. It contains a superconvergence result for the pressure.

A priori estimate for the postprocessed pressure

It holds

$$|||p - p_h^*|||_{t,h} \leq C\Big((t+h)h^k\|\mathbf{u}\|_{k+1} + (t+h)^{-1}h^{k+2}\|p\|_{k+2}\Big).$$

The postprocessing is crucial for the a posteriori estimate.

A posteriori estimate

The same as before with the addition of the jumps in velocity and pressure

$$\|\mathbf{u} - \mathbf{u}_h\|_t + |||p - p_h^*|||_{t,h} \leq C\Big(\sum_{K \in \mathcal{C}_h} E_K(\mathbf{u}_h, p_h^*)^2 \Big)^{1/2}$$

$$E_K(\mathbf{u}_h, p_h)^2 = \frac{h_K^2}{t^2 + h_K^2} \|t^2 \Delta \mathbf{u}_h - \mathbf{u}_h - \nabla p_h^* + \mathbf{f}\|_{0,K}^2$$
$$+ (t^2 + h_K^2)\|\mathrm{div}\, \mathbf{u}_h - g\|_{0,K}^2$$
$$+ \frac{h_K}{t^2 + h_K^2} \|[\![t^2 \partial_n(\mathbf{u}_h)]\!]\|_{0,\partial K \setminus \partial\Omega}^2 + \frac{t^2 + h_K^2}{h_K} \|(\mathbf{u}_h - \mathbf{u}_D) \cdot \mathbf{n}\|_{0,\partial K \cap \partial\Omega}^2$$
$$+ \frac{t^2}{h_E} \|[\![\mathbf{u}_h]\!]\|_{0,\partial K} + + \frac{h_E}{t^2 + h_E^2} \|[\![p_h^*]\!]\|_{0,\partial K}.$$

The last slide!

- A general a priori and a posteriori analysis.
  Three families of methods.
- Theory: CALCOLO and to appaer on our wegpage.
  Numerical results: see web page.
- Previous work on numerics: A priori with $H(\mathrm{div})$-norms by Tai, Mardal, Winther 2002 and Badia, Codina 2009.
- MERCI.