Convergence of Generalized Volume Averaging Method on a Convection-Diffusion Problem

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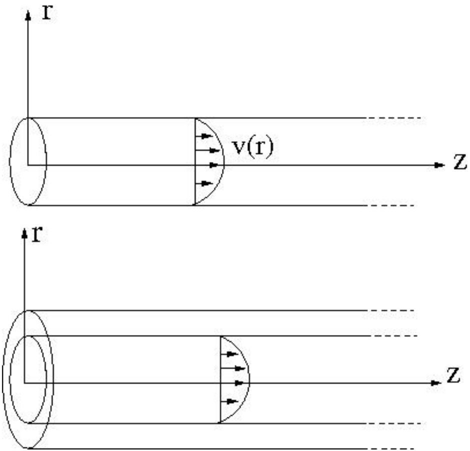
Financial support from GDR MOMAS

4-5 Novembre 2008

Problems at ends



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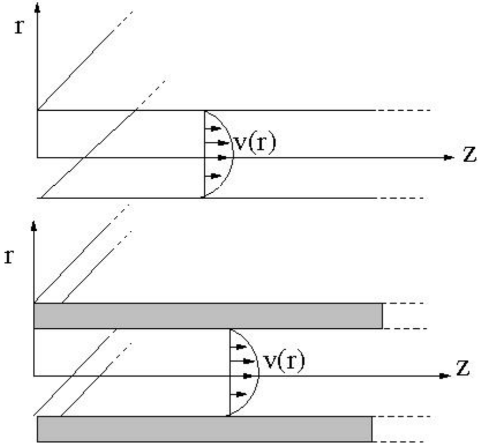

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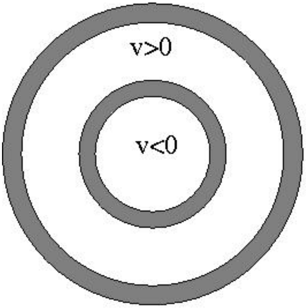

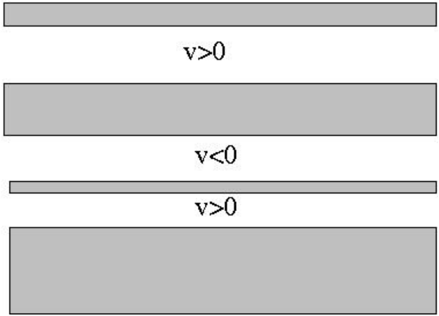

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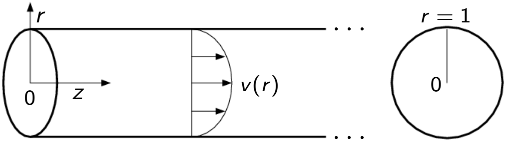
- Volume averaging

Back to Graetz problem

Unit radius tube
Axi-symmetry
Large Péclet $Pe \gg 1$.



Taylor Approximation $\rightarrow$ axial diffusion negligeable $O(1/Pe^2)$

Directionnal problem $\rightarrow$ A entry condition is given $T_0(r)$,

$$\frac{1}{r}\partial_r\left(r\,\partial_r T\right) = Pe\ v(r)\,\partial_z T \quad , \quad T(r,0) = T_0(r) \quad , \quad T(1,z) = 0 \ .$$

Eigen-function expansion $\rightarrow$ $T = \sum_{\lambda\in\Lambda} c_\lambda t_\lambda(r) e^{\lambda z}$

Eigenvalue problem $\rightarrow$ $\lambda,\ t_\lambda(r)$

$$\Delta_c t_\lambda = \lambda\ Pe\ v(r)\ t_\lambda \quad , \quad t_\lambda(1) = 0 \ .$$

Averaging Graetz problem

- Average temperature $T^\star(z)$

$$T^\star \;=\; \int_0^1 T(r,z)\; r\; dr$$

- Searching for a 1-D Macroscopic equation for $T^\star(z)$
- Usual decomposition $T(r,z) = T^\star(z) + \theta(r,z)$
- Average temperature fulfills

$$\langle \Delta_c T \rangle^\star = Pe\partial_z \langle vT \rangle^\star$$

- Deviation fulfills

$$\Delta_c\theta - \langle \Delta_c\theta \rangle^\star \equiv \mathcal{L}^\star\theta = (v - \langle v \rangle^\star)\, Pe\partial_z T^\star + Pe\partial_z \left(v\theta - \langle v\theta \rangle^\star\right)$$

Averaging Graetz problem

- Closure relation :

$$T^\star(z) + \theta = \sum_n \alpha_n(r)\partial_z^n T^\star(z)$$

Averaging Graetz problem

- Similar property for the exact solution ?

$$T(r,z) = \sum_n a_n(r)\partial_z^n T^\star(z)$$

- Since,

$$T(r,z) = \sum_{\lambda\in\Lambda} c_\lambda t_\lambda(r) e^{\lambda z}$$

- Then

$$t_\lambda(r) = \sum_n a_n(r)\,\lambda^n$$

- λ-analyticity of the solution $\Rightarrow$ Validity of closure relation

λ-analyticity of Graetz solution ?

- Graetz solution :
- $t_\lambda(r) = \exp^{-\lambda r^2/2} M(\frac{1}{2} - \frac{\lambda}{4}, 1, \lambda r^2)$
- $M(a, b, x)$: Confluent Hypergeometric function

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- $t_\lambda(r)$ is analytic in λ

Computing the averaged description

- closure problem for α_n

$$\left\{ \begin{array}{ccc} \mathcal{L}^\star \alpha_0(r) & = & 0 \\ \alpha_0^\star & = & 1 \end{array} \right.$$

$$\left\{ \begin{array}{ccccc} \mathcal{L}^\star \alpha_n & = & v(r)\alpha_{n-1}(r) - \langle v\alpha_{n-1}\rangle^\star & \text{with} & \alpha_{-1}(r) = 0 \\ \alpha_0^\star = 1 & \text{or} & \alpha_n^\star = 0 & \text{for} & n \geq 1 \\ \alpha_n(1) & = & 0 & \text{for} & \mathcal{D} \\ \partial_r \alpha_n(1) & = & 0 & \text{for} & \mathcal{N} \end{array} \right.$$

- Macroscopic 1-D equation :

$$\sum_{n=0} K_n Pe^n \partial_z^n T^\star(z) = 0 \quad ,$$

$$K_n = \langle \Delta_c \alpha_n \rangle^\star - \langle v\alpha_{n-1}\rangle^\star \ , \quad K_n \in \mathbb{R},$$

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- 4. Generalizations ?

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- "Averaged" Spectrum Λ_p :

$$\lim_{p \to \infty} \Lambda_p = \left\{ \lambda \quad / \quad \sum_{n=0}^{p} K_n \lambda^n = 0 \right\} \quad .$$

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Convergence theorem for Graetz problem : $\Lambda_p \cap \Lambda$ is not empty

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- $D^\star_{acc}$ depends on the averaging (weight function)
- Compute $D^\star_{acc}$ for various situations for Graetz problem in Pierre et al., SIAP, **66**, (2006)

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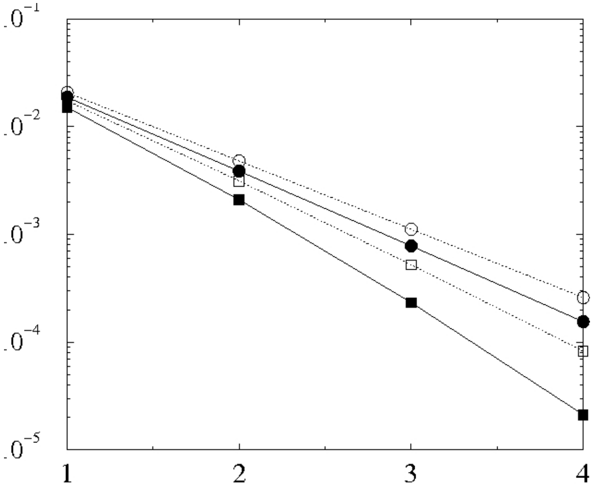

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- With additionnal physical effects ?