



Fluid/solid coupled convection/diffusion in unidirectional flows.

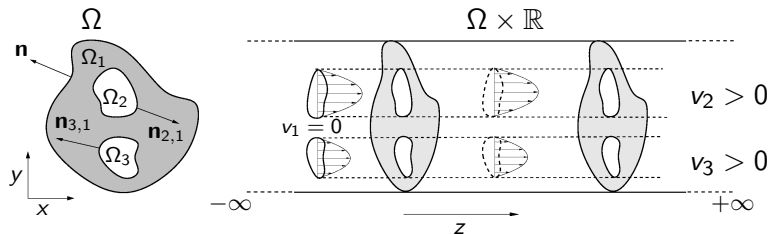
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5 novembre 2008

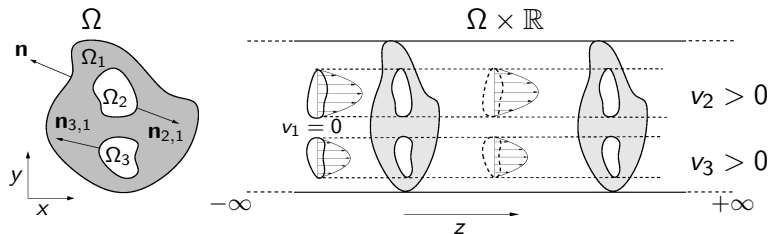
1- The problem : physical configuration



Heat transfer in an infinite cylinder with cross-section Ω .

- 3 sub-domains : Ω_1 (solid), $\Omega_{2,3}$ (fluid).
- Laminar steady flow : $\vec{\mathbf{v}} = v(x, y)\vec{\mathbf{e}}_z$.
- $v_i = v|_{\Omega_i}$, here : $v_1 = 0$ (solid), $v_2, v_3 \neq 0$ (fluid).
- Heterogeneous conductivities k : $k_i = k|_{\Omega_i}$, $k_i \neq k_j$.
- $\Gamma_{i,j}$ interface between Ω_i and Ω_j , $\mathbf{n}_{i,j}$ normal to $\Gamma_{i,j}$.

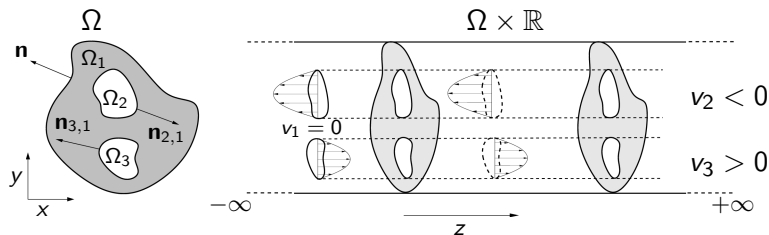
1- The problem : physical configuration



This settlement both include :

- co-current flows,
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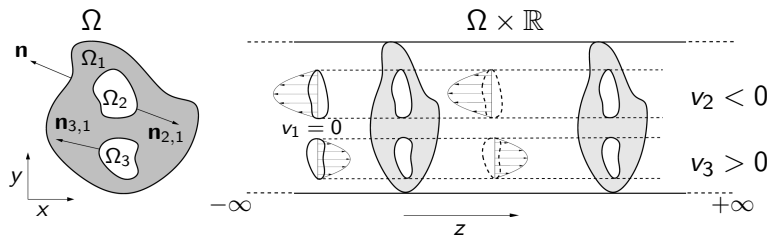
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1- The problem : physical configuration



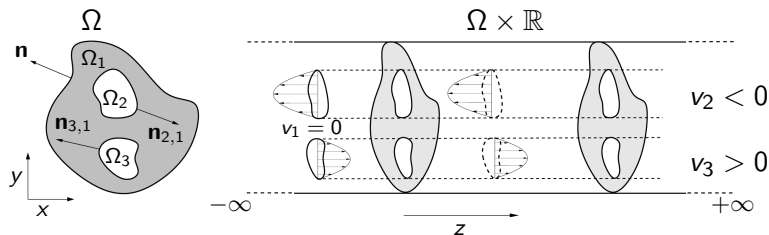
This settlement both include :

- co-current flows,
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Possible extensions to :

- planar configurations (unbounded in x),
- periodic configurations.

2- The problem : mathematical formulation



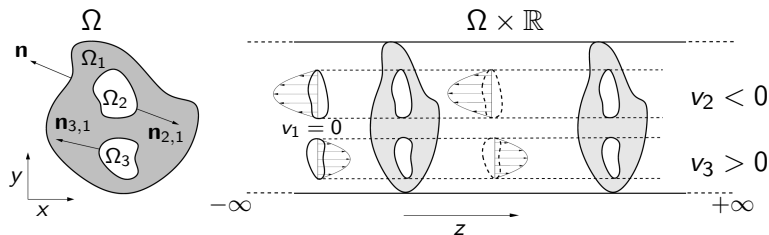
Energy equation on the temperature T : $v = v(x, y)$, $k = k(x, y)$,

$$\operatorname{div}(k \nabla T) + k \partial_z^2 T = \operatorname{Pe} v \partial_z T,$$

+ Continuity coupling conditions between the sub-domains :

$$T_i = T_j, \quad k_i \nabla T_i \cdot \mathbf{n}_{i,j} = k_j \nabla T_j \cdot \mathbf{n}_{i,j} \quad \text{on } \Gamma_{i,j},$$

2- The problem : mathematical formulation



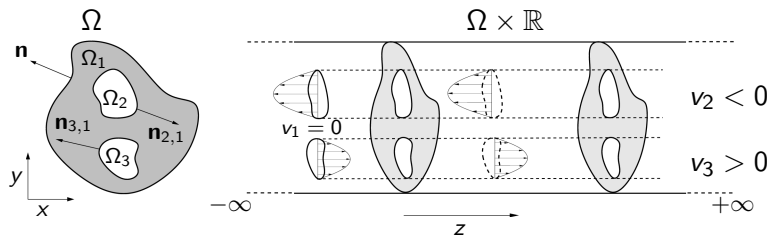
Energy equation on the temperature T : $v = v(x, y)$, $k = k(x, y)$,

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- + Boundary conditions on $\partial\Omega$: Dirichlet with jump at $z = 0$,
- + Limit conditions at $\pm\infty$:

$$\text{on } \partial\Omega : \begin{cases} T = 1, & z < 0 \\ T = 0, & z > 0 \end{cases} \quad \text{and} \quad \begin{cases} T \rightarrow 1, & z \rightarrow -\infty \\ T \rightarrow 0, & z \rightarrow +\infty \end{cases}.$$

3- Objectives



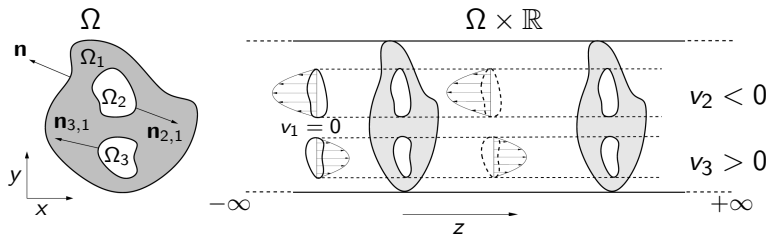
1. Macroscopic description of *an average temperature* $T^*(z)$:

$$T^*(z) \simeq C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} + \dots$$

2. Exchanges between sub-domains description :

$$\int_{\Gamma_{i,j}} k_i \nabla T_i \cdot \mathbf{n}_{i,j} ds .$$

4- Pending questions



1. Does T read : $T(x, y, z) = \sum_{\lambda \in \Lambda} c_\lambda t_\lambda(x, y) e^{\lambda z}$?
2. Location of the "spectrum" Λ , get a computation method for the *eigenvalues/eigenfunctions* λ , $t_\lambda(x, y)$.
3. Computation of the constants c_λ : searching an *orthogonality property* for the t_λ .

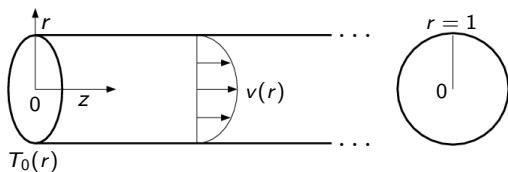
Introductory exemple : the Graetz problem

Semi-infinite tube (radius 1)

1 fluid phase

Axi-symmetry

High Péclet : $Pe \gg 1$



Taylor approximation \rightarrow axial diffusion $\partial_z^2 T$ neglected

"Directional" problem \rightarrow entry condition $T_0(r)$ given

$$\frac{1}{r} \partial_r (r \partial_r T) = v(r) \partial_z T, \quad T(r, 0) = T_0(r), \quad T(1, z) = 0.$$

Separate variable $\rightarrow T = t(r) e^{\lambda z}$

Eigenvalue problem $\rightarrow \lambda, t(r)$ read :

$$\frac{1}{r} \partial_r (r \partial_r t) = \lambda Pe v(r) t, \quad t(1) = 0.$$

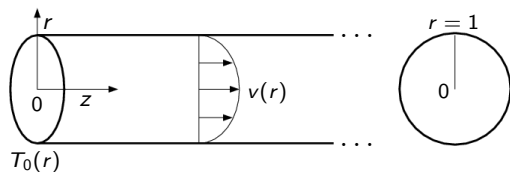
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Self-adjoint, negative and compact problem :

\Rightarrow Complete orthogonal system of eigenfunctions $(t_i(r))_i$,
with eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \dots \rightarrow -\infty$.

\Rightarrow Analytical solution :

$$T(r, z) = \sum_{i \in \mathbb{N}} c_i t_i(r) e^{\lambda_i z} \quad , \quad c_i = \int_0^1 t_i T_0 r dr \quad .$$

Generalisation 1 : extended Graetz

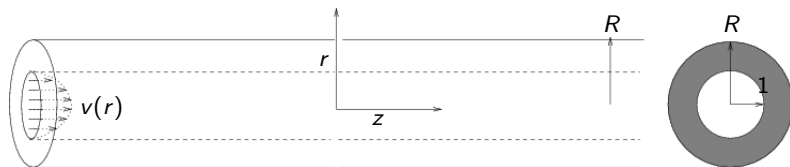
Axial diffusion $\partial_z^2 T$ is no longer neglected.

1. Separate variable $\rightarrow T = t(r) e^{\lambda z}$
 \rightarrow do not provide an eigenvalue problem

$$\frac{1}{r} \partial_r (r \partial_r t) = (\lambda v(r) - \lambda^2) t .$$

2. No symmetry property available :
 \rightarrow problem for the spectrum location : $\Lambda \in \mathbb{R}$?
 \rightarrow computational problem for the c_λ .
3. The problem is not *directional* any more :
entry condition $T_0(r)$ not relevant.
 \rightarrow switch to limit conditions in $\pm\infty$.

Generalisation 2 : conjugated Graetz



Coupling with a solid wall where diffusion occurs.

$$1 < r < R : \frac{1}{r} \partial_r (r \partial_r T) + \partial_z^2 T = 0$$

$$r = 1 : T(1^+, z) = T(1^-, z), \quad \partial_r T(1^+, z) = k \partial_r T(1^+, z).$$

⇒ **Same difficulties as before :**

1. no *real* eigenvalue problem,
2. no symmetry property ,
3. problem not *directional*.

Mixed reformulation : statement

One reformulate the initial problem

$$\operatorname{div}(k\nabla T) + k\partial_z^2 T = v\partial_z T ,$$

adding a vectorial unknown $\mathbf{X} = \mathbf{X}(x, y, z)$:

$$k \partial_z T = v T - \operatorname{div}(\mathbf{X})$$

$$\partial_z \mathbf{X} = k \nabla T$$

→ Introducing the operator A :

$$\partial_z \begin{vmatrix} T \\ \mathbf{X} \end{vmatrix} = A \begin{vmatrix} T \\ \mathbf{X} \end{vmatrix} , \quad A = \begin{pmatrix} v k^{-1} & -k^{-1} \operatorname{div} \\ k \nabla & \end{pmatrix}$$

Mixed reformulation : analysis

Theorem 1. The unbounded operator $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$,

$$\mathcal{H} = L^2(\Omega) \times L^2(\Omega)^2 \quad , \quad D(A) = H_0^1(\Omega) \times H(\operatorname{div}, \Omega) \quad :$$

1. is self adjoint,
2. is diagonal on an eigenfunctions orthogonal system,
3. $\lambda_0 = 0$ excepted, all eigenvalues have finite order.

Its spectrum Λ reads

$$\Lambda = \{\lambda_0\} \cup \Lambda^+ \cup \Lambda^- \quad :$$

- Λ^+ = **downstream** modes : $0 > \lambda_1^+ \geq \lambda_2^+ \geq \dots \rightarrow -\infty$
 \rightarrow related to the $z > 0$ region.
- Λ^- = **upstream** modes : $0 < \lambda_1^- \leq \lambda_2^- \leq \dots \rightarrow +\infty$
 \rightarrow related to the $z < 0$ region.

Mixed reformulation : solution definition

Analytical solution : defined from

- Downstream eigenvalues / eigenfunctions : λ_n^+ , $t_n^+(x, y)$
- Upstream eigenvalues / eigenfunctions : λ_n^- , $t_n^-(x, y)$
- The coefficients α_n

$$\alpha_n := \frac{1}{\lambda_n^2} \int_{\partial\Omega} k \nabla t_n \cdot \mathbf{n} \, ds ,$$

Corollary . The sought temperature field reads :

$$T(x, y, z) = \begin{cases} 1 + \sum_n \alpha_n^- t_n^-(x, y) e^{\lambda_n^- z} & z \leq 0 \\ - \sum_n \alpha_n^+ t_n^+(x, y) e^{\lambda_n^+ z} & z \geq 0 \end{cases}$$

Some numerical analysis

The following eigen-problem has to be solved :

$$\text{find } \lambda \in \mathbb{R}, \quad \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. \in D(A) : \quad A \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. = \lambda \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. . \quad (1)$$

Theorem 2. Eigen-problem (1) is equivalent to the following variational problem :

find $\lambda \in \mathbb{R}$ and $(T, \mathbf{X}) \in L^2(\Omega) \times H(\text{div}, \Omega)$,

such that $\forall (u, \mathbf{Y}) \in L^2(\Omega) \times H(\text{div}, \Omega) :$

$$\begin{aligned} \int_{\Omega} T u v \, dx - \int_{\Omega} u \text{div}(\mathbf{X}) \, dx &= \lambda \int_{\Omega} T u k \, dx \\ - \int_{\Omega} T \text{div}(\mathbf{Y}) \, dx &= \lambda \int_{\Omega} \mathbf{X} \cdot \mathbf{Y} k^{-1} \, dx . \end{aligned}$$

Axi-symmetric convergence analysis

Discretisation using mixed finite element spaces :

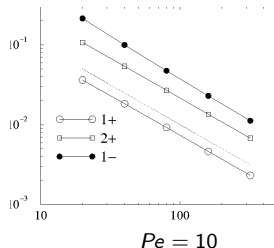
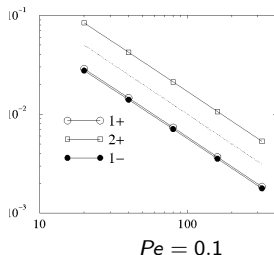
$$e. g. T_h \in \mathbb{P}^0, \mathbf{X}_h \in RT_0.$$

Evaluation of the method on the conjugated Graetz problem :

- Reduction to a 1D numerical problem (axi-symmetry).
- Comparison with analytical reference solutions.

Relative error on the first eigenfunctions :
 T_1^+ , T_2^+ and T_1^- ,
 with respect to
 the nodes number.

Dashed line : slope = -1



Convergence rate on the eigenvalues

→ same order 1.

Conclusion

- *Nice mathematical framework* : orthogonality properties, problem analysis on a complete orthogonal base.
- Natural mixed numerical formulation.
- From 3D to 2D problem reduction,
Only smallest modulus eigenvalues to be computed (principal modes),
Numerical validation on a test case.

Conclusion and perspectives

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- Natural mixed numerical formulation.
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- General 2D implementation.

- Study of fracture/fracture coupling in diluted fractures networks.