

Convection-diffusion equations in random and locally periodic media

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Introduction

This talk will focus on macroscopic description of convection-diffusion processes in media with a microstructure. We will mainly study the following model problems:

- **Convection-diffusion problem in a non-stationary periodic medium.** The mathematical description of the long-term behaviour of convection-diffusion processes in the medium with a microstructure whose properties may vary in time, relies on the following parabolic problem

$$(1) \quad \partial_t u^\varepsilon - \operatorname{div} \left(a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon \right) - \frac{1}{\varepsilon} b \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon = f(x, t)$$

$$u^\varepsilon|_{t=0} = u_0(x)$$

with a small positive parameter ε which characterizes the microscopic length scale of the medium. We will consider two model problems, namely

1. **Periodic both in temporal and spatial variables coefficients;**
2. **Periodic in spatial variables and random stationary ergodic in temporal variable coefficients.**

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- **Nonlinear model.** The corresponding Cauchy problem reads

$$\partial_t u^\varepsilon - \operatorname{div} \left(a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon \right) - \frac{1}{\varepsilon} b \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u^\varepsilon \right) \nabla u^\varepsilon = f(x, t)$$

$$u^\varepsilon|_{t=0} = u_0(x).$$

We will show that the dispersion effect takes place. Namely, the effective (homogenized) diffusion contains a nonlinear term.

- **Rapidly pulsating perforation.** We will consider the simplest model problem for the standard heat equation and assume that the homogeneous Neumann condition is imposed on the perforation boundary. The microscopic geometry is spatially periodic.

Two cases of pulsation will be considered:

1. **Periodic in time pulsation;**
2. **Random stationary in time pulsation.**

Introduction

A convection-diffusion equation of the form

$$\partial_t u^\varepsilon - \operatorname{div} \left(a \left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right) - b \left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon = f(x, t)$$

and the corresponding stationary equation

$$\operatorname{div} \left(a \left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right) + b \left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon = f(x)$$

can be studied by the classical homogenization methods. However, the above mentioned equations with large convection terms are more realistic in mechanical and physical models. Indeed, if we want to describe the long-term behaviour of a convection-diffusion process in a given periodic or random medium, we should study the large time asymptotics of solutions to the following equation

$$\partial_s u^\varepsilon - \operatorname{div} \left(a(y, s) \nabla u^\varepsilon \right) - b(y, s) \nabla u = f.$$

If we denote the large time by T , then $\varepsilon = \frac{1}{\sqrt{T}}$ is a small parameter. Making the change of variables $x = \varepsilon y$, $t = \varepsilon^2 s$, we arrive at equation (1).

Previously, the convection-diffusion equations with zero effective drift were studied in many mathematical works, for example the operators with periodic coefficients were treated in the book of A. Bensoussan, J.-L. Lions, G. Papanicolaou. The operators with divergence free random convection were studied by M. Avellaneda, A. Majda and S. Kozlov. A. Fannjang, G. Papanicolaou and V. Zhikov considered operators with divergence free convection and small diffusion.

The case of periodic potential vector field was treated in the works of S. Kozlov and A. Piatnitski.

An equation with nonzero effective drift was also studied in the recent work of G. Allaire.

Closely related problems were studied by T. Goudon and F. Poupaud

Periodic non-stationary medium

Convection-diffusion problem in periodic medium

$$(2) \begin{cases} \partial_t u^\varepsilon - \operatorname{div} \left(a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u \right) - \frac{1}{\varepsilon} b \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u = 0, & \text{in } (0, T) \times \mathbb{R}^d, \\ u(x, 0) = u_0(x). \end{cases}$$

Assumptions:

- $a_{ij}(y, s), b_j(y, s) \in L^\infty(\mathbb{R}^d), i, j = \overline{1, d}$;
- $\exists \Lambda > 0: a_{ij}(y, s) \xi_i \xi_j \geq \Lambda |\xi|^2, \quad \forall s \in \mathbb{R}, \quad y, \xi \in \mathbb{R}^d$;
- $a_{ij}(y, s), b_j(y, s)$ are $[0, 1]^{d+1}$ -periodic functions; in other words, all the coefficients are 1-periodic in time and in each spatial coordinate direction.
- $u_0(x) \in L^2(\mathbb{R}^d)$.

Auxiliary constructions

Adjoint periodic problem. Consider the differential operators

$$\mathcal{A}v = \partial_s v - \operatorname{div}(a(y, s)\nabla v) - b(y, s)\nabla v$$

and

$$\mathcal{A}^*v = -\partial_s v + \operatorname{div}(a(y, s)\nabla v) - \operatorname{div}(b(y, s)v)$$

Lemma 1 *The equation $\mathcal{A}^*p = 0$ has a unique (up to a multiplicative constant) $[0, 1]^{d+1}$ -periodic solution $p(y, s)$. Under the normalization*

$$\int_{[0,1]^{d+1}} p(y, s) dy ds = 1$$

$p(y, s)$ satisfies the following bounds

$$0 < p^- \leq p(y, s) \leq p^+ < \infty.$$

Lemma 2 *Let $f(y, s)$ be a $[0, 1]^{d+1}$ -periodic function. Then the equation $Av = f(y, s)$ has a $[0, 1]^{d+1}$ -periodic solution $v(y, s)$ if and only if*

$$\int_{[0,1]^{d+1}} f(y, s)p(y, s) dyds = 0.$$

This solution is unique up to an additive constant.

Denote

$$\bar{b} = \int_{[0,1]^{d+1}} (\operatorname{div} a(y, s) + b(y, s))p(y, s) dyds.$$

Later we will show that the vector $\frac{1}{\varepsilon}\bar{b}$ determines the effective convection in the studied problem.

Homogenization result

We introduce moving coordinates $(x, t) \mapsto (x - \frac{\bar{b}}{\varepsilon}t, t)$ and denote $v^\varepsilon(x, t) = u^\varepsilon(x - \frac{\bar{b}}{\varepsilon}t, t)$.

Theorem 1 *The function v^ε converges, as $\varepsilon \rightarrow 0$, in $L^2((0, T) \times \mathbb{R}^d)$ to a solution of the following Cauchy problem:*

$$\partial_t u^0 - \operatorname{div} \left(\hat{a} \nabla u^0 \right) = 0,$$

$$u^0(x, 0) = u_0(x),$$

where

$$\hat{a} = \int_{[0,1]^{d+1}} \left(\mathbf{I} + \nabla \chi(y, s) \right)^T a(y, s) \left(\mathbf{I} + \nabla \chi(y, s) \right) p(y, s) dy ds,$$

and $\chi(y, s)$ is a periodic solution of the equation

$$(3) \quad \mathcal{A}\chi(y, s) = -\operatorname{div} a(y, s) - b(y, s) + \bar{b}.$$

Remark 1 *Equation (3) is solvable due to the definition of \bar{b} .*

Random case (joint work with M. Kleptsyna)

We turn to the equation with random in time coefficients.

$$(4) \quad \begin{cases} \partial_t u^\varepsilon - \operatorname{div} \left(a \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u \right) - \frac{1}{\varepsilon} b \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u = 0, & \text{in } (0, T) \times \mathbb{R}^d, \\ u(x, 0) = u_0(x). \end{cases}$$

Assumptions:

- $a_{ij}(y, s), b_j(y, s) \in L^\infty(\mathbb{R}^d)$, $i, j = \overline{1, d}$;
- $\exists \Lambda > 0: a_{ij}(y, s) \xi_i \xi_j \geq \Lambda |\xi|^2$, $\forall s \in \mathbb{R}$, $y, \xi \in \mathbb{R}^d$;
- $a_{ij}(y, s), b_j(y, s)$ are $[0, 1]^d$ -periodic functions of y and random stationary ergodic functions of time;
- The strong mixing coefficient $\alpha(r)$ of $(a(y, s), b(y, s))$ satisfies the following condition:

$$\int_0^\infty \sqrt{\alpha(r)} dr < \infty;$$

- $u_0(x) \in L^2(\mathbb{R}^d)$.

Strong mixing coefficient

Denote by $\mathcal{F}_{\leq t}$ the σ -algebra generated by $\{a(y, s), b(y, s), s \leq t\}$. Similarly we define $\mathcal{F}_{\geq t}$.

Definition 1 *The function $\alpha(r)$ defined by*

$$\alpha(r) = \sup_{\substack{A \in \mathcal{F}_{\leq 0}, \\ B \in \mathcal{F}_{\geq r}}} \left(\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B) \right)$$

is called the strong mixing coefficient.

Auxiliary result

Recall the notations

$$\mathcal{A}v = \partial_s v - \operatorname{div}(a(y, s)\nabla v) - b(y, s)\nabla v$$

and

$$\mathcal{A}^*v = -\partial_s v + \operatorname{div}(a(y, s)\nabla v) - \operatorname{div}(b(y, s)v).$$

Lemma 3 *The equation $\mathcal{A}^*p(y, s) = 0$ has a unique, up to a multiplicative constant, periodic in y and stationary in s solution. Under the normalization $\int_{[0,1]^d} p(y, s) dy = 1$ this solution satisfies the estimate:*

$$0 < p^- \leq p(y, s) \leq p^+ < \infty.$$

We introduce the constant vector

$$\bar{b} = \mathbf{E} \int_{[0,1]^d} \left(\operatorname{div} a(y, s) + b(y, s) \right) p(y, s) dy;$$

and the stationary random process

$$\eta(s) = \int_{[0,1]^d} \left(\operatorname{div} a(y, s) + b(y, s) - \bar{b} \right) p(y, s) dy.$$

Remark 2 *The vector \bar{b} does not depend on s because all the processes taking part in its definition are stationary.*

Also we introduce two matrices:

$$\{\Lambda^2\}_{ij} = \int_0^\infty \mathbf{E} \left(\eta_i(s) \eta_j(0) + \eta_i(0) \eta_j(s) \right) ds;$$

$$\hat{a} = \mathbf{E} \int_{[0,1]^d} \left(\mathbf{I} + \nabla \chi(y, s) \right)^T a(y, s) \left(\mathbf{I} + \nabla \chi(y, s) \right) p(y, s) dy,$$

where χ is a stationary solution of the equation

$$(5) \quad \mathcal{A}\chi(y, s) = \operatorname{div} a(y, s) + b(y, s) - \bar{b} - \eta(s).$$

Lemma 4 *Equation (5) has a unique, up to an additive constant, stationary solution in the space of $[0, 1]^d$ -periodic in y functions.*

Homogenization

Consider the following stochastic PDE:

$$(6) \quad du(t) = \left(\hat{a}_{ij} + \frac{1}{2} (\Lambda^2)_{ij} \right) \frac{\partial^2}{\partial x_i \partial x_j} u(t) dt + \Lambda \nabla_x u(t) dW_t,$$

$$u(x, 0) = u_0(x),$$

where W_t is the standard d -dimensional Wiener process. As in the case of the periodic coefficients, we introduce moving coordinates and denote $v^\varepsilon(x, t) = u^\varepsilon(x - \frac{\bar{b}}{\varepsilon}t, t)$.

Theorem 2 *The functions v^ε converge in law in the space $L^2((0, T) \times \mathbb{R}^d)$, as $\varepsilon \rightarrow 0$, to a solution of (6).*

Remark 3 *It should be noted that the pointwise convergence result might fail to hold in the random case. The above theorem only states the convergence in law.*

Nonlinear case (joint work with E. Marusic-Paloka)

Nonlinear convection-diffusion problem.

$$\partial_t u - \operatorname{div} \left(a \left(\frac{x'}{\varepsilon} \right) \nabla u^\varepsilon \right) - \varepsilon^{-1} b \left(\frac{x'}{\varepsilon}, u^\varepsilon \right) \nabla u^\varepsilon = 0,$$

$$u^\varepsilon(x, 0) = u_0(x),$$

where $x = (x', x_d)$ and $b(y', v)$ has a special structure:

$$b(y', v) = \left(b_1(y'), \dots, b_{d-1}(y'), b_d(y')g(v) \right).$$

Assumptions. We impose the standard uniform ellipticity and periodicity assumptions. For simplicity we also assume that all the coefficients, $u_0(x)$ and the function g are sufficiently smooth.

Denote $a'(y') = a_{ij}(y')$, $i, j = 1, \dots, d-1$, $b'(y') = (b_1(y'), \dots, b_{d-1}(y'))$ and effective drift

$$\bar{b}' = \int_{[0,1]^{d-1}} \left(\operatorname{div}_{y'} a'(y') + b'(y') \right) p'(y') dy',$$

where $p'(y')$ is a solution of the equation

$$\operatorname{div}_{y'} (a'(y') \nabla_{y'} p') - \operatorname{div}_{y'} (b'(y') p') = 0.$$

Again we introduce moving coordinates, and set

$$v^\varepsilon(x, t) = u^\varepsilon(x - \varepsilon^{-1} \bar{b}' t, t).$$

Homogenization result

Theorem 3 Assume that $\int_{[0,1]^{d-1}} b_d(y') p'(y') dy' = 0$. Then v^ε converges, as $\varepsilon \rightarrow 0$, in $L^2([0, T] \times \mathbb{R}^d)$ to a solution u^0 of the following homogenized problem:

$$\partial_t u^0 - \operatorname{div} \left(\hat{a}(u^0) \nabla u^0 \right) = 0,$$

$$u^0(x, 0) = u_0(x),$$

where

$$\hat{a}(u^0) \xi \cdot \xi = \int_{[0,1]^{d-1}} p'(y') a(y') \nabla_{y'} w \cdot \nabla_{y'} w dy',$$

$$w = (\chi' + y') \cdot \xi + g(u^0) \theta(y') \xi_d,$$

and the functions χ' and θ solve the following equations:

$$\operatorname{div}_{y'} \left(a'(y') \nabla_{y'} \chi' \right) + b'(y') \nabla_{y'} \chi' = -\operatorname{div}_{y'} a'(y') - b'(y') + \bar{b}',$$

$$\operatorname{div}_{y'} \left(a'(y') \nabla_{y'} \theta \right) + b'(y') \nabla_{y'} \theta = -b_d(y').$$

Pulsating perforation.

The geometry Let F_t be a family of diffeomorphisms of \mathbb{R}^d onto itself, $t \in \mathbb{R}$.

We assume that for any integer vector $z \in \mathbb{Z}^d$:

$$F_t(x + z) = F_t(x) + z, \quad \forall x \in \mathbb{R}^d$$

and that F_t is 1-periodic in t .

Denote $B_0 = \{|x| \leq 1/4\}$ and $B = \bigcup_{z \in \mathbb{Z}^d} (B_0 + z)$.

We introduce $G(t) = F_t(\mathbb{R}^d \setminus B)$, then $G(t)$ is 1-periodic.

Domain with pulsating perforation.

$$Q_T^\varepsilon = \left\{ (x, t) \in \mathbb{R}^{d+1} : 0 \leq t < T, \frac{x}{\varepsilon} \in G\left(\frac{t}{\varepsilon^2}\right) \right\}.$$

Homogenization problem

Consider the following problem

$$\partial_t u^\varepsilon - \Delta u^\varepsilon = 0 \quad \text{in } Q_T^\varepsilon$$

$$\frac{\partial u^\varepsilon}{\partial n^\varepsilon} = 0 \quad \text{on } \partial Q_T^\varepsilon, \quad u^\varepsilon(x, 0) = u_0(x);$$

n^ε is the external normal in x -variable.

Cell problems

First cell problem reads

$$\partial_s p(y, s) + \Delta_y p(y, s) = 0 \text{ in } G$$

$$\frac{\partial p}{\partial n_z} + n_s p = 0 \text{ on } \partial G,$$

n_z and n_s are the components of the $(d + 1)$ -dimensional normal.

Lemma 5 *There is a unique, up to a multiplicative constant, (y, s) -periodic solution $p(y, s)$.*

Under proper normalization it satisfies the estimate $0 < p^- \leq p(y, s) \leq p^+ < \infty$.

Second cell problem

$$\partial_s \chi - \Delta_y \chi = -\bar{b} \text{ in } G,$$

$$\frac{\partial \chi}{\partial n} = -n(y, s) \text{ on } \partial G$$

with

$$\bar{b} = \int_{\partial G} p(y, s) n(y, s) ds_y ds.$$

Homogenization result

Lemma 6 *There is a unique, up to an additive constant, (y, s) -periodic solution $\chi(y, s)$.*

Denote

$$v^\varepsilon(x, t) = u^\varepsilon\left(x - \frac{\bar{b}}{\varepsilon}t, t\right).$$

Theorem 4 *The function v^ε converges, as $\varepsilon \rightarrow 0$, to a solution $u^0(x, t)$ of the following problem:*

$$\partial_t u^0 - \operatorname{div}(\hat{a} \nabla u^0) = 0,$$

$$u^0(x, 0) = u_0(x).$$