Discretization of diffusion equations on general grids

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joint work with R. Eymard and T. Gallouët

R. Eymard, T. Gallouët, and R. Herbin. A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis. C. R., Math., Acad. Sci. Paris, 344(6):403406, 2007.

R. Eymard and R. Herbin. A new colocated finite volume scheme for the incompressible Navier-Stokes equations on general non matching grids. C. R. Math. Acad. Sci. Paris, 344(10):659662, 2007.

Diffusion equation

Ω is an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$.

$$-\mathrm{div}(\Lambda(x)\nabla u) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

Λ measurable such that $\Lambda(x)$ symmetric and $Sp(\Lambda(x)) \subset [\underline{\lambda}, \overline{\lambda}]$.
Weak formulation:
Find $u \in H_0^1(\Omega)$;

$$\int_\Omega K(x)\nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x)v(x)\, dx, \ \forall v \in H_0^1(\Omega).$$

Aim

Discretization of the diffusion problem with two constraints:

- "General" meshes (any polygonal cells, non conforming cells)
- sparse s.d.p. matrices
- If possible, cheap scheme : one unknown per cell.

with at least:

- Existence and uniqueness of the approximate solution
- Convergence result of the approximate solution to the exact solution when the mesh size goes to 0.

First approximate problem (1)

$\mathcal{T}$ is a mesh of Ω.
$H_{\mathcal{T}}$ is the set of functions $u : \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

First idea:

For $u \in H_{\mathcal{T}}$, define $\nabla_{\mathcal{T}} u$ (convenient approximation of the gradient on u)

Approximate problem :

$$\int_{\Omega} K \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

Definition of the approximate solution on edges

For $K \in \mathcal{T}$, let $x_K \in \mathcal{T}$ (K is "x_K star shaped").
For σ, interior edge (interface) of $\mathcal{T}$, x_σ is the **center** of σ.

$$x_\sigma = \sum_{M \in \mathcal{T}} a_{M,\sigma} x_M.$$

For $u \in H_\mathcal{T}$,

$$\Pi_\sigma u = \sum_{M \in \mathcal{T}} a_{M,\sigma} u_M.$$

If σ is an edge on the boundary, one sets $\Pi_\sigma u = 0$.

Definition of the approximate gradient

For $K \in \mathcal{T}$,

$$\frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma n_{K,\sigma} (x_\sigma - x_K)^t = Id.$$

For $u \in H_\mathcal{T}$, approximate gradient, $\nabla_\mathcal{T} u$ on K :

$$\nabla_K u = (\nabla_\mathcal{T} u)_K \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\Pi_\sigma u - u_K) n_{K,\sigma}.$$

First approximate problem (2)

Problem: $\nabla_{\mathcal{T}} u = 0 \not\Rightarrow u = 0$

First approximate problem :

$$\int_\Omega \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx = \int fv \, dx, \; \forall v \in H_{\mathcal{T}}.$$

$\rightsquigarrow$ no existence, no uniqueness, no coercivity on $H_{\mathcal{T}}$...

Stabilization using consistency estimate, $R_{\mathcal{T}} u$

$u \in H_{\mathcal{T}}$, $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, the value of $R_{\mathcal{T}} u$ is on the cone $D_{K,\sigma}$:

$$R_{K,\sigma} u = \frac{\alpha_K}{d_{K,\sigma}} (\Pi_\sigma u - u_K - \nabla_K u \cdot (x_\sigma - x_K)).$$

If u is a regular function $u_K = u(x_K)$, $R_{K,\sigma} u \to 0$ as the mesh size goes to 0.

Approximate problem

$H_{\mathcal{T}}$ is the set of functions $u : \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

$$u \in H_{\mathcal{T}},$$

$$\int_\Omega \Lambda \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx + b(u,v) = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

$$b(u,v) = \int_\Omega R_{\mathcal{T}} u \, R_{\mathcal{T}} v \, dx$$

main properties:

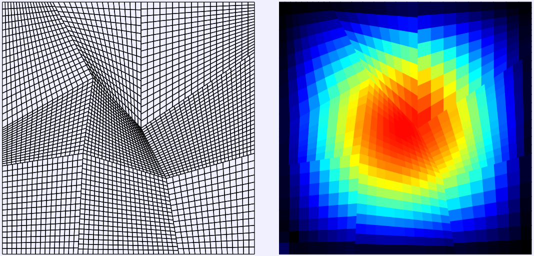
- Coercivity of the discrete operator, existence of a solution and estimate
- Monotony of b, uniqueness of the solution
- $b(u,u) \geq 0$
- Convenient bound on u, and v such that $v_K = \psi(x_K)$ with ψ regular ($v = P_{\mathcal{T}} \psi$) gives $b(u,v) \to 0$ as the $\mathrm{size}(\mathcal{T}) \to 0$

Numerical example, rotating anisotropy

Example from C. Le Potier (CEA)
$\Omega = [0,1]^2, \tilde{x}_i = x_i + 0.1, \ i = 1,2, \varepsilon > 0$

$$\Lambda(x) = \begin{pmatrix} \tilde{x}_2^2 + \varepsilon \tilde{x}_1^2 & -(1-\varepsilon)\tilde{x}_1\tilde{x}_2 \\ -(1-\varepsilon)\tilde{x}_1\tilde{x}_2 & \tilde{x}_1^2 + \varepsilon \tilde{x}_2^2 \end{pmatrix}$$

Eigenvalues $\underline{\lambda}(x) = \varepsilon(\tilde{x}_1^2 + \tilde{x}_2^2), \overline{\lambda}(x) = \tilde{x}_1^2 + \tilde{x}_2^2$ with $\tilde{x}_1^2 + \tilde{x}_2^2 \in [0.02, 2.42]$
Exact solution $u(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$ for f well chosen.



Finite volume scheme ?

On super-admissible meshes: $x_\sigma - x_K = d_{K,\sigma}\mathbf{n}_{\mathbf{K},\sigma}$ (triangles, rectangles, rectangular parallelepipeds)

Taking $\alpha_K = \sqrt{d}$ and

$$\Pi_\sigma u = \frac{d_{L,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_K + \frac{d_{K,\sigma}}{d_{K,\sigma}+d_{L,\sigma}} u_L \text{ if } \mathcal{T}_\sigma = \{K, L\}$$

$\rightsquigarrow$ two points FV scheme

$$\int_\Omega \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v dx =$$

$$\sum_{\substack{\sigma\in\Omega \\ \mathcal{T}_\sigma=\{K,L\}}} \frac{m_\sigma}{d_{K,\sigma}+d_{L,\sigma}}(u_L - u_K)(v_L - v_K) + \sum_{\substack{\sigma\subset\partial\Omega \\ \mathcal{T}_\sigma=\{K}} \frac{m_\sigma}{d_{K,\sigma}} u_K v_K$$

On general meshes flux between K and L s.t. $\exists \sigma \in \mathcal{E}_K,\ a_\sigma^L \neq 0$ or $\exists \sigma \in \mathcal{E}_L,\ a_\sigma^K \neq 0$

No local interface flux conservativity $\rightsquigarrow$ problems when coupling to transport equation in highly heterogeneous media.

Finite volume hybrid scheme

Unknowns: $U = \{(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\in\mathcal{E}}\}$
Same definition of the gradient and bilinear form,
but now u_σ is not determined by a barycentric formula but by local conservativity of the fluxes:

$$F_{K,\sigma}(U) = -F_{L,\sigma}(U)$$

where $F^{\mathcal{D}}_{K,\sigma}(U)$ approximation of $-\int_\sigma \Lambda_K \nabla \bar{u}(x) \cdot \mathbf{n}_{\mathbf{K},\sigma} \mathrm{d}\mathbf{s}(\mathbf{x})$, with

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma) = \int_\Omega \Lambda \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx + b(u, v)$$

Excellent for heterogeneous media, but expensive scheme!

Compromise: semi hybrid scheme

Principle:

Determine u_σ by local conservativity of the flux on heterogenous interfaces

Determine u_σ by a barycentric formula from u_K elsewhere.

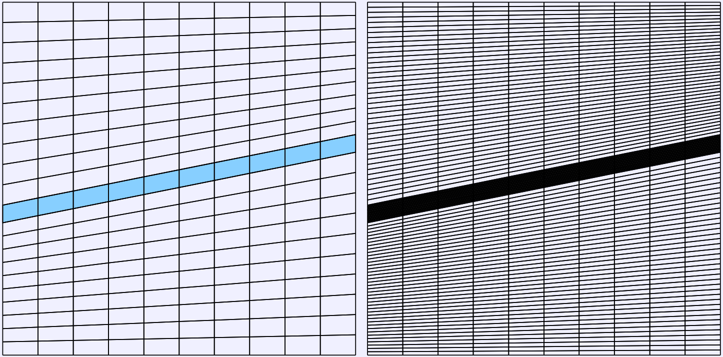


Figure: Meshes used for the tilted barrier test: mesh 1 (10×21 left), mesh 2 (10×100 right)

$\Lambda(x) = \lambda(x)\mathrm{Id}$, with $\lambda(x) = 1$ for a.e. $x \in \Omega_1$, $\lambda(x) = 10^{-2}$ for a.e. $x \in \Omega_2$ and $\lambda(x) = 1$ for a.e. Ω_3.

Approximations of the four fluxes at the boundary.

	nunk	nmat	$x = 0$	$x = 1$	$y = 0$	$y = 1$
exact			-0.2	0.2	1.	-1.
PB-M1	210	2424	-1.17	1.17	3.51	-3.51
PB-M2	1000	11904	-0.296	0.296	1.27	-1.27
PB-M3	250	2904	-0.208	0.208	1.02	-1.02
BH-M1	239	2583	-0.2	0.2	1.	-1.
PH-M1	599	4311	-0.2	0.2	1.	-1.

PB = Pure barycentric
PH = Pure hybrid
BH = Hybrid on heterogeneous interfaces

M1 coarse mesh
M2 fine mesh
M3 coarse mesh + two fine layers of cells along each interface

Conclusions

Scheme suitable for general meshes, Highly heterogeneous media, 2 or 3D complex geometries.

Reasonable number of unknowns

Proof of convergence and error estimate for regular solutions.

Easily generalizable for non linear problems (convergence proven for the p Laplacian) and incompressible Navier Stokes equations.

Perspectives

Comparison with other schemes: 2D benchmark FVCA5 June 9 13 Aussois (google FVCA5)

3D benchmark, coupling with transport equation.

Preservation of the physical bounds (discrete maximum principle). How to deal with negative transmissivities ?

Convergence for non regular right hand side (Dirac)