Adaptive Aitken-Schwarz Domain Decomposition for flow in porous media

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GDR MOMAS Thema C, November, 14 2006
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Outline

- Extension of the Aitken-Schwarz domain decomposition in the case of heterogeneous non separable operator.
- Introduce adaptivity to compute the acceleration matrix
- 3D Benchmark on flow in porous media

Outline

Acceleration of Schwarz Method for Linear Separable Operators

Garbey & DTD, *On some Aitken like acceleration of the Schwarz method*, Int. J. for Numerical Methods in Fluids, 40(12) :1493-1513,2002

- 1D Jacobi Schwarz algorithm for linear separable operators :

$$\left\{\begin{array}{ll} L[u_1^{n+1}] = f \ in\ \Omega_1, & u^{n+1}_{1|\Gamma_1} = u^n_{2|\Gamma_1} \\ L[u_2^{n+1}] = f \ in\ \Omega_2, & u^{n+1}_{2|\Gamma_2} = u^n_{1|\Gamma_2} \end{array}\right.$$

- the interface error operator P is **linear**, i.e

$$\left\{\begin{array}{lll} u^{n+1}_{1|\Gamma_2} - U_{|\Gamma_2} & = & \delta_1(u^n_{2|\Gamma_1} - U_{|\Gamma_1}) \\ u^{n+1}_{2|\Gamma_1} - U_{|\Gamma_1} & = & \delta_2(u^n_{1|\Gamma_2} - U_{|\Gamma_2}) \end{array}\right. \Rightarrow \quad P = \begin{pmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{pmatrix}$$

- Consequently

$$\left\{\begin{array}{lll} u^2_{1|\Gamma_2} - u^1_{1|\Gamma_2} & = & \delta_1(u^1_{2|\Gamma_1} - u^0_{2|\Gamma_1}) \\ u^2_{2|\Gamma_1} - u^1_{2|\Gamma_1} & = & \delta_2(u^1_{1|\Gamma_2} - u^0_{1|\Gamma_2}) \end{array}\right.$$

- Computation of $\delta_{1/2}$:
  $L[v_{1/2}] = 0 \ in\ \Omega_{1/2}$, $v_{\Gamma_{1/2}} = 1$. thus $\delta_{1/2} = v_{\Gamma_{2/1}}$.
- iff $\delta_1\delta_2 \neq 1$ **Aitken-Schwarz** gives the solution with exactly 3 iterations and possibly 2 in the analytical case.

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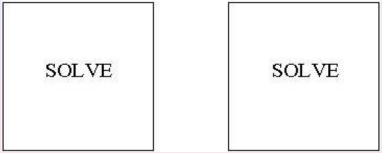
At interfaces Γ_1 and Γ_2, the Fourier coefficients of the error of Jacobi Schwarz algorithm can be rearranged on the form :

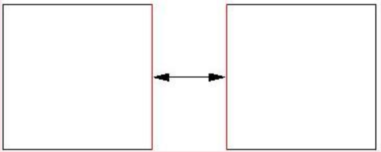
$$\hat{e}_1^{(n+2)}(\Gamma_1) = P\hat{e}_1^{(n)}(\Gamma_1)$$
$$\hat{e}_2^{(n+2)}(\Gamma_2) = P\hat{e}_2^{(n)}(\Gamma_2)$$

For a separable operator in 2D or 3D :

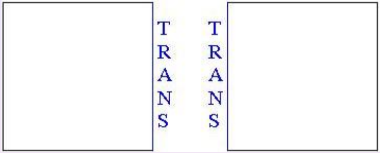
- step1 : build *P* analytically or numerically from data given by two Schwarz iterates
- step2 : apply one Jacobi Schwarz iterate to the differential problem with block solver of choice i.e multigrids, FFT etc...



- step3 : exchange boundary information :



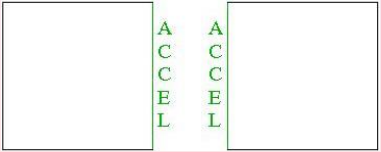
- step4 : compute the Fourier expansion $\hat{u}^n_{j|\Gamma_i}, n = 0, 1$ of **the traces on the artificial interface** $\Gamma_i, i = 1..nd$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the Schwarz iterate solution $u^1_{|\Gamma_i}$.



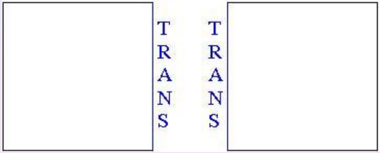
- step5 : apply generalized Aitken acceleration based on

$$\hat{u}^\infty = (Id - P)^{-1}(\hat{u}^1 - P\hat{u}^0)$$

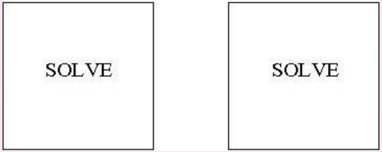
in order to get $\hat{u}^\infty_{|\Gamma_i}$.



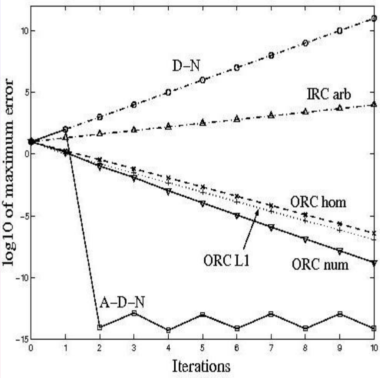
- step6 : recompose the trace $u^{\infty}_{|\Gamma_i}$ in physical space through the Inverse Fourier Transform :



- step7 : compute in parallel the solution in each subdomain Ω_j, with new inner BCs and blocksolver of choice.



Numerical results for uniform meshes



$$\begin{cases} -k_1 \Delta u_1 & = & f \text{ in } \Omega_1 \\ -k_2 \Delta u_2 & = & f \text{ in } \Omega_2 \end{cases}$$

(see Calugaru & Tromeur-Dervout, *Non-Overlapping DDMs of Schwarz type to Solve Flow in Discontinuous Porous Media*, in Domain Decomposition Methods in Science and Engineering, 40 :529–536,2004)

Artificial interface conditions : Dirichlet-Neumann (D-N), arbitrary Robin (IRC), optimized order 0 Robin (ORChom) (see Nataf & al.,*Optimal Interface Conditions for Domain Decomposition Methods*, Technical Report Number 301, CMAP 301, 1994), optimized order 0 Robin (ORCL1, numerical optimization), and Aitken Dirichlet-Neumann (A-D-N)

Possible extensions for non-uniform interface meshes :

1. Projection technique : spectral Chebycheff interpolation of the interface traces on a third regular grid
   + classical AS with FFT
   Boursier,Tromeur-Dervout and Vassilevsky, *Parallel solution of Mixed Finite Element/ Spectral Element systems for convection-diffusion equations on non matching grids*,Preprint CDCSP-0300, 2004
   Periodic extension after Chebycheff interpolation ⇒ coupled Fourier modes
   Not taken into account when constructing P.

2. Baranger, Garbey and Oudin-Dardun *Generalized Aitken-like acceleration of the Schwarz method*, Lecture Notes in Computational Science and Engineering, pages 505-512, 2004

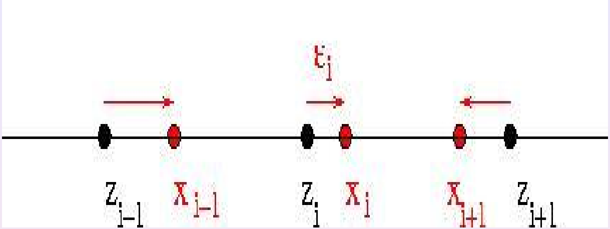
- Approximate in the physical space P with band matrix P^*.
- Compute eigenvalues and eigenvectors of matrix P^*.
- Take eigenvectors as “generalized Fourier” basis functions

No available tool to know how the band width of the approximate P^* can give eigenvalues close to the eigenvalues of true P.

Outline

Non Uniform Fourier Transform formulation

- Gridding : interpolation + FFT on an oversampled grid $\Rightarrow$ approximation dependent on the spreading distance from the nonuniform mesh Greengard and Lee, *Accelerating the Nonuniform Fast Fourier Transform*, SIAM REVIEW, vol.46, No.3, pp.443-454, 2004
- Aim : benefit of the Fourier decomposition and the exponential decrease of Fourier modes
  - Define a set of basis functions $\Phi_l = (\phi_l(x_j))_{0 \leq j \leq N}$ strictly related to the nonuniform mesh and orthogonal with respect to a sesquilinear form $[[.,.]]$, i.e $[[\phi_l, \phi_k]] = 0, \text{ if } l \neq k$.
  - Compute the associated interface operator $P_{[[.,.]]}$
  - Approximate $P_{[[.,.]]}$ with $P^*_{[[.,.]]}$ through a posteriori estimates of Fourier coefficients behavior.



Definition

Let $(x_i)_{0 \leq i \leq N}$ and $z_i = \frac{2\pi i}{N}$ such that $x_i = z_i + \epsilon_i$, and

$$\phi_l(x) = \begin{cases} \psi_l(x) = \exp(ilx), \ 0 \leq l \leq N/2 \\ D^{-N} \exp(i(N-l)x), \ N/2+1 \leq l \leq N, \\ D = diag(\epsilon_i)_{0 \leq i \leq N} \end{cases} \tag{1}$$

$\Rightarrow \phi_{N-l}(x) = \overline{\phi_l(x)}$.

- Uniform mesh : $f \in L^2(0, 2\pi) \Rightarrow$ its Fourier series :

$$\Pi^F f = \sum_{k=-\infty}^{\infty} \hat{f}_k \phi_k, \quad \text{with } \phi_k(x) = e^{ikx},$$

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} (f, \phi_k).$$

- Discrete scalar product $\Rightarrow$ discrete Fourier coefficients :

$$(f, g)_N = h \sum_{j=0}^{N-1} f(x_j) \overline{g(x_j)} \quad \Rightarrow \quad \tilde{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) e^{i(k-\frac{N}{2})x_j}$$

- Discrete Fourier series of f of order N :

$$\Pi_N^F f(x) = \sum_{k=0}^{N-1} \tilde{f}_k e^{i(k-\frac{N}{2})x}$$

$\Rightarrow$ Generalize discrete scalar product to non uniform mesh to guarantee :

- basis function orthogonality
- interpolation of the trigonometric polynomial $\Pi_N^F f$ at points x_j :

$$\Pi_N^F f(x_j) = f(x_j), \qquad j = 0, 1, \cdots, N-1. \tag{2}$$

Definition

Define sesquilinear form on $S_N = \text{span}\{\phi_l(x), 0 \leq l \leq N\}$, using Hermite integration formula :

$$[[f,g]] = \sum_{l=0}^{N} \gamma_l f(x_l)\overline{g(x_l)} + \sum_{l=0}^{N} \beta_l (f'(x_l)\overline{g(x_l)} + f(x_l)\overline{g'(x_l)})$$

$\{\gamma_l\}$ and $\{\beta_l\}$: $[[\phi_l, \phi_k]] = \delta_{lk} \Rightarrow$ solve one L.S. (size 2N)
H $=([[\phi_l, \phi_k]])_{l,k=0,\ldots N} = Id \Rightarrow$ [[:, :]] Hermitian on $S_N = \text{span}\{\phi_l(x), 0 \leq l \leq N\}$

Definition

The discrete Fourier coefficients of f are given by :

$$\tilde{f}_k = [[f, \Phi_k]], \quad k = -N/2, ..., N/2$$

$$\tilde{f} = M_1 f + M_2 f', \quad M_1, M_2 \in \mathcal{M}_{N+1}(\mathbb{C})$$

$$M_1(k,l) = \gamma_l \overline{\phi_k(x_l)} + \beta_l \overline{\phi'_k(x_l)},\ M_2(k,l) = \beta_l \overline{\phi_k(x_l)}$$

Proposition

$$\Pi_N^F(f(x)) = \sum_{k=0}^{N} \tilde{f}_k \phi_k(x), \quad \textit{is exact}\ \forall f \in \mathbb{T}^{N/2}([0,2\pi[)$$

Proof

$$f \in \mathbb{T}^{N/2}([0,2\pi[) \Rightarrow f(z) = \sum_{k=-N/2}^{N/2} \eta_k \exp(ikz) = \sum_{m=0}^{N} \eta_m \Phi_m(z).$$

$$\tilde{f}_k = [[f,\Phi_k]] = \sum_{m=0}^{N} \eta_m [[\Phi_m,\Phi_k]] = \sum_{m=0}^{N} \eta_m \delta_{mk} = \eta_k$$

$$\Rightarrow \Pi_N^F(f(z)) = \sum_{k=0}^{N} \tilde{f}_k \phi_k(z) = \sum_{k=0}^{N} \eta_k \Phi_k(z) = f(z).$$

- Problem : in the applications one is given the vector f which represents the values of a function $f(x)$ on the points $(x_i)_{0 \leq i \leq N}$. No information on the vector f', needed in the Hermitian form definition.
- 1st approach : approximate f' through an order 6 Taylor development
  $\Rightarrow$ solve a size 6 LS for each point
- 2nd solution : determine f' implicitly by imposing

$$\frac{d}{dx}(\Pi_N^F(f(x)))_{|x=x_l} = f'(x_l), \quad l = 0, ..., N-1$$

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In an algebraic form, if we note M_ϕ the matrix whose elements are :

$$M_\phi(l,k) = \phi'_k(x_l)$$

then the vector f' is obtained by solving the algebraic system :

$$(id_{N+1} - M_\phi M_2)f' = M_\phi M_1 f$$

where id_N is the identity matrix in $\mathcal{M}_{N+1}(\mathbb{C})$.

- Given a nonuniform mesh $(x_i)_{0 \leq i \leq N}$, define the basis functions and solve one L.S. (size 2N) to determine the two sets $\{\gamma_l\}$ and $\{\beta_l\}$.
- Solve the algebraic system (size N) :

$$(id_{N+1} - M_\phi M_2)f' = M_\phi M_1 f$$

  to determine f' implicitly.
- Compute Fourier coefficients through matrix-vector products :

$$\tilde{f} = M_1 f + M_2 f'$$

- For uniform grids NUDFT $\rightarrow$ classical Discrete Fourier Transform

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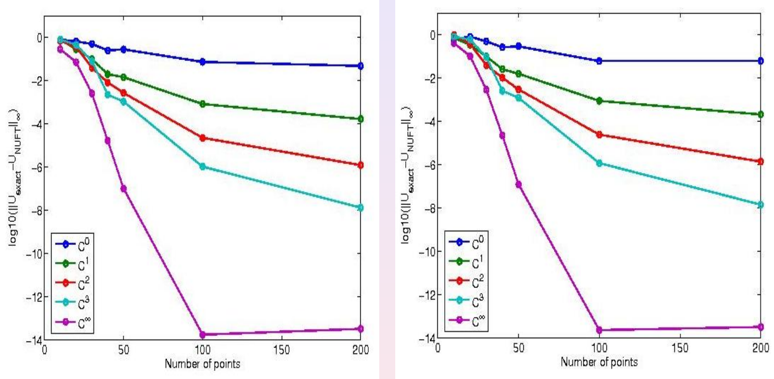
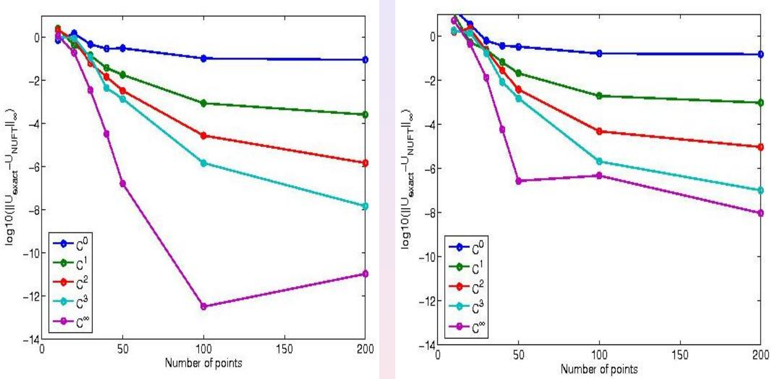
N	$\varepsilon = h_u/8$	$\varepsilon = h_u/4$	$\varepsilon = h_u/2$	$\varepsilon = h_u$
40	0.13E-14 6.17E+3	0.39E-15 1.21E+4	0.56E-13 1.26E+5	0.62E-7 4.24E+10
100	0.77E-14 8.40E+4	0.17E-14 1.82E+5	0.69E-12 1.25E+6	0.83E-7 2.07E+10
200	0.13E-13 6.02E+5	0.16E-13 1.27E+6	0.27E-12 5.75E+6	0.5E-6 9.35E+11
400	0.26E-13 5.18E+6	0.29E-13 1.24E+7	0.11E-10 2.96E+8	0.53E-8 7.30E+10

TAB.: Interpolation properties of the 1D NUDFT : $\|f - \Pi_N^F(f)\|_\infty$ and $cond_2([[.,.]])$ for $f(x) = \exp(-40(x-(2\pi/3))^2)$, with $h_u = 2\pi/N$.

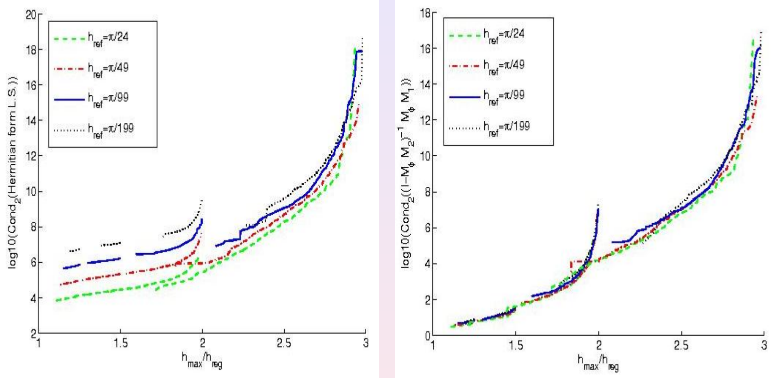


Approximation properties of the 1D NUDFT, as a fonction of the solution continuity, with $\epsilon = h/8$ (left) and $\epsilon = h/4$ (right).



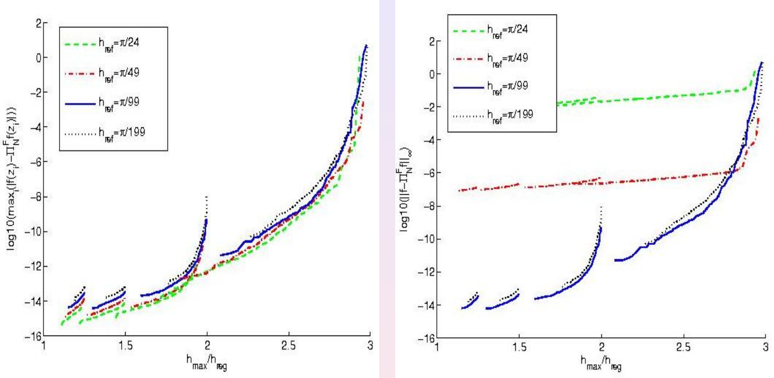
Approximation properties of the 1D NUDFT, as a fonction of the solution continuity, with $\epsilon = h/2$ (left) and $\epsilon = h$ (right).

Numerical results



Condition number of the linear system involved in the computation of the Hermitian form (left) and of the derivatives (right).

Numerical results



NUDFT interpolation property (left) and approximation property for a C^{∞} function (right) as a function of the ratio between the irregular grid and the regular one.

- Given a nonuniform Cartesian 2D mesh $\mathbf{x} \times \mathbf{y} := \{(x_i, y_j)_{0 \le i,j \le N}\} \subset \mathbb{R}^2$ define the basis functions, the sesquilinear form :

$$[[f,g]] = \sum_{j=0}^{N} \gamma_j \Big(\sum_{l=0}^{N} \alpha_l (f\overline{g})(x_j, y_l) + \sum_{l=0}^{N} \eta_l \partial_y (f\overline{g})(x_j, y_l) \Big) + \sum_{j=0}^{N} \beta_j \Big(\sum_{l=0}^{N} \alpha_l \partial_x (f\overline{g})(x_j, y_l) + \sum_{l=0}^{N} \eta_l \partial_{xy} (f\overline{g})(x_j, y_l) \Big)$$

- Fourier coefficients computed algebraically by previously solving implicitly for $\partial_x f, \partial_y f$ and $\partial_{xy} f$.

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\begin{aligned}
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& \sum_{j=0}^{N} \beta_j \Big(\sum_{l=0}^{N} \alpha_l \partial_x (f\overline{g})(x_j, y_l) + \sum_{l=0}^{N} \eta_l \partial_{xy} (f\overline{g})(x_j, y_l) \Big)
\end{aligned}
$$

- Fourier coefficients computed algebraically by previously solving implicitly for $\partial_x f, \partial_y f$ and $\partial_{xy} f$.

N	$\epsilon = h_u/2$	$\epsilon = h_u$	$\epsilon = 2h_u$	$\epsilon = 4h_u$
2^7	1.1E-13 1.5E+3	3.7E-13 8E+3	9.5E-7 2.5E+6	2.09E+3 2.2E+12
2^8	2.62E-13 6E+3	1.48E-10 5E+5	8E-4 1.7E+10	3E+6 1E+14

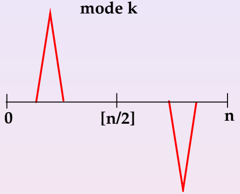
TAB.: Interpolation properties of the 2D NUDFT : $\|f - \Pi_N^F(f)\|_\infty$ and $cond_2([[.,.]])$ for $f(x,y) = \cos^2(x)\cos(y)$, with $h_u = 2\pi/N$.

At interfaces Γ_1 and Γ_2, the Fourier coefficients of the error of Jacobi Schwarz algorithm can be rearranged on the form :

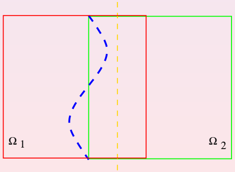
$$\hat{e}_1^{(n+2)}(\Gamma_1) = P_{[[.,.]]}\hat{e}_1^{(n)}(\Gamma_1)$$
$$\hat{e}_2^{(n+2)}(\Gamma_2) = P_{[[.,.]]}\hat{e}_2^{(n)}(\Gamma_2)$$

Numerically, $P_{[[.,.]]}$ is computed by applying two Schwarz iterates for each Fourier mode of the interface solution (computed through the NUDFT), as a relation between all the modes at the two iterates.

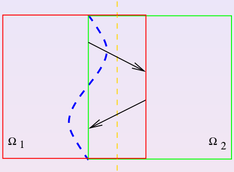
- Take one Fourier mode on the interface :



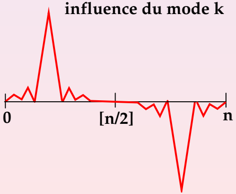
- Apply inverse NUDFT (blue line) :



- With 2 Schwarz iterates determine how this function is modified by the Jacobi Schwarz algorithm :

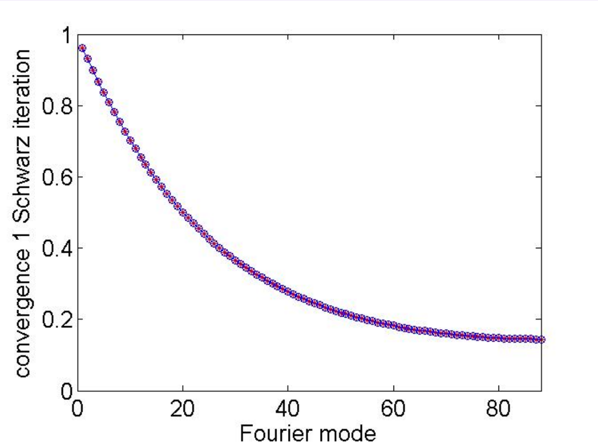


- Apply NUDFT and compute the influence of one Fourier mode on all modes :



- Fill $k-$column of matrix $P_{[[.,.]]}$, not symmetric.

Uniform grids and Poisson problem : NUDFT $\rightarrow$ FFT
$P_{[[.,.]]}$ diagonal and $\|P_{[[.,.]]} - P_{an}\|_\infty = O(10^{-12})$



Advantages of Non Uniform Discrete Fourier Transform

- Advantages :
  - better performance than FFT on nonuniform meshes when applied to Aitken-Schwarz DDM
  - $O(N^2)$ operations $\rightarrow$ cheaper in time in comparison with the $O(N^3)$ operations to solve for the eigenvalues and eigenvectors of the full interface operator
  - Adaptive approximation of the trace transfer operator $P_{[[.,.]]}$, based on a posteriori error estimates of Fourier modes convergence

Outline

Nonuniform Cartesian grids and/or non separable differential operator $\Rightarrow$ P is no longer diagonal

- Select Fourier modes higher than a fixed tolerance and through their decreasing factor between 2 Schwarz iterations. Index = array containing the list of selected modes.
- Take the subset $\tilde{v}$ of Fourier modes from 1 to max(Index).
- Approximate $P_{[[.,.]]}$ with $P^*_{[[.,.]]}$ using only $\tilde{v}$.
- Accelerate $\tilde{v}$ through the equation :

$$\tilde{v}^{\infty} = (Id - P^*_{[[.,.]]})^{-1}(\tilde{v}^{n+1} - P^*_{[[.,.]]}\tilde{v}^{n})$$

  Other modes are not accelerated.
- $P^*_{[[.,.]]}$ columns can be built in parallel and the number of columns computed during the Schwarz iterates can be set according to the computer architecture

DTD
Aim
AS
NUDFT
NUDFT + AS
Adaptive AS
On going Work on the benchmark

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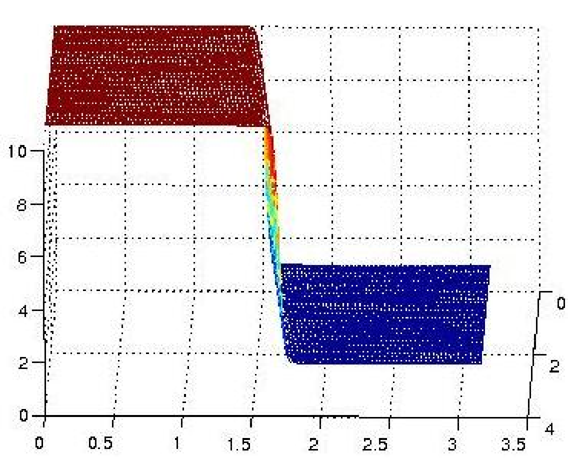
$$\begin{cases} \nabla.(a(x,y)\nabla)u(x,y) = f(x,y), & \text{on } \Omega =]0,1[^2 \\ u(x,y) = 0, & (x,y) \in \partial\Omega \end{cases}$$

$a(x,y) = a_0 + (1-a_0)(1 + tanh((x - (3h * y + 1/2 - h))/\mu))/2,$
and $a_0 = 10^1, \mu = 10^{-2}$.



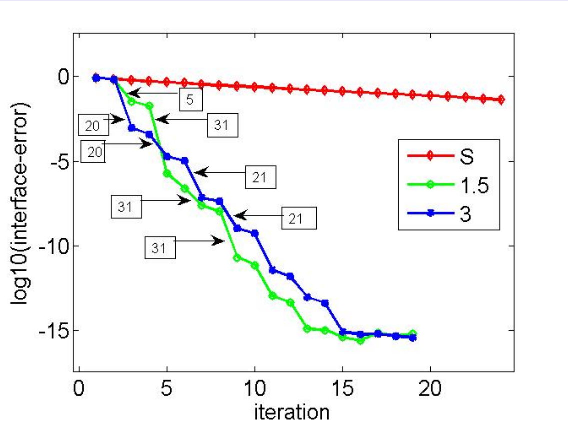


FIG.: adaptive acceleration using sub-blocks of $P_{[[.,.]]}$, with 100 points on the interface, overlap$=1$, $\epsilon = h_u/8$ and Fourier modes tolerance = $||\hat{u}^k||_\infty/10^i$ for $i = 1.5$ and 3 for 1st iteration and $i = 4$ for successive iterations.

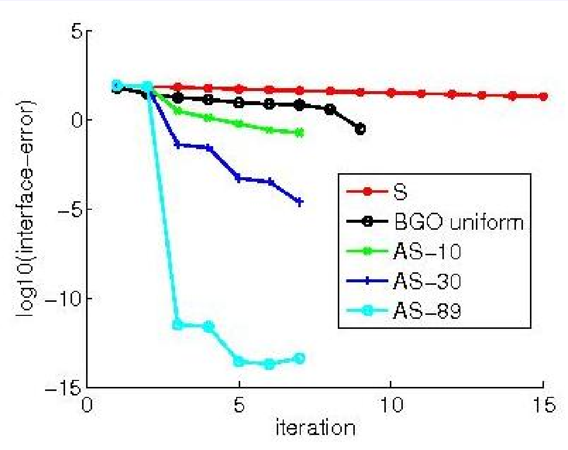
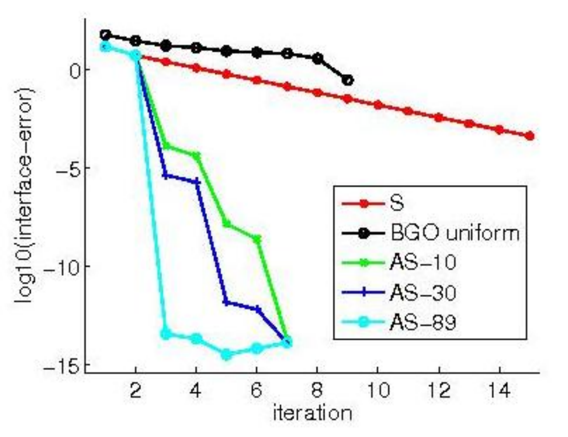


FIG.: acceleration using sub-blocks of $P_{[[.,.]]}$ with 90 points on the interface, overlap$=1$ and $\epsilon = h_u/2$. Black line refers to results for a uniform grid and overlap=5 in Baranger & al., *The Aitken-Like Acceleration of the Schwarz Method on Non-Uniform Cartesian Grids*, Technical Report Number UH-CS-05-18, 2005.



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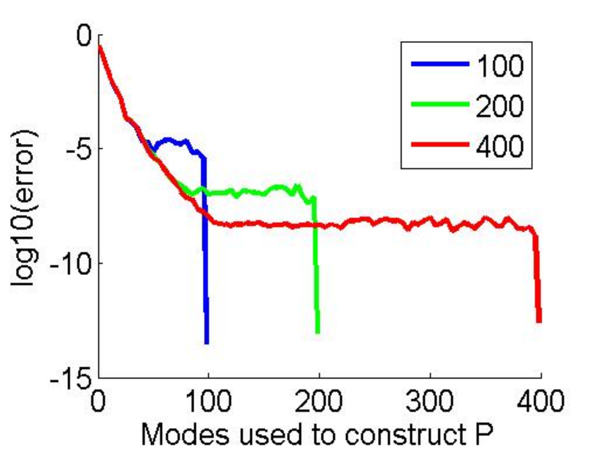
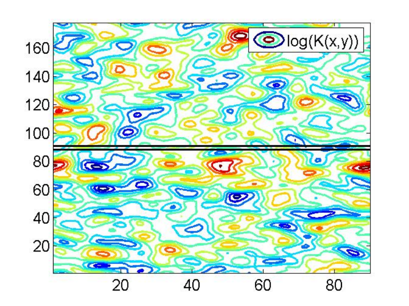


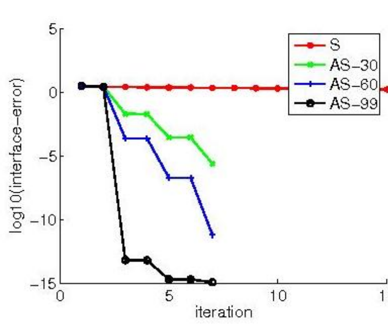
FIG.: influence of the approximation of the interface operator $P_{[[.,.]]}$ on the convergence of the interface error, for $\epsilon = h_u/2$.

K follows a log-normal random process

$$\nabla.(K(x,y)\nabla u) = f, \ \text{on}\Omega$$
$$u = 0, \ \text{on } \partial\Omega$$

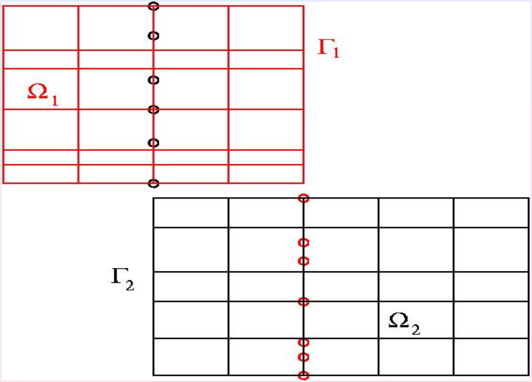


$K(x,y) \in [0.0091, 242.66]$



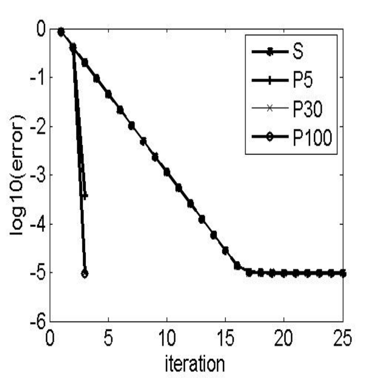
Convergence of AS

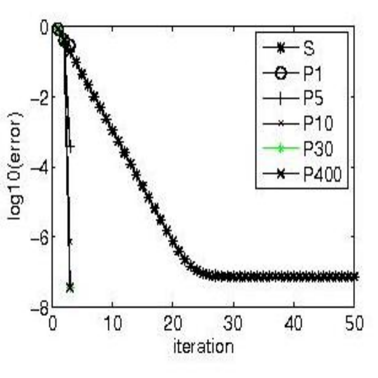
Work in collaboration with J-R De Dreuzy and J. Erhel SAGE/IRISA



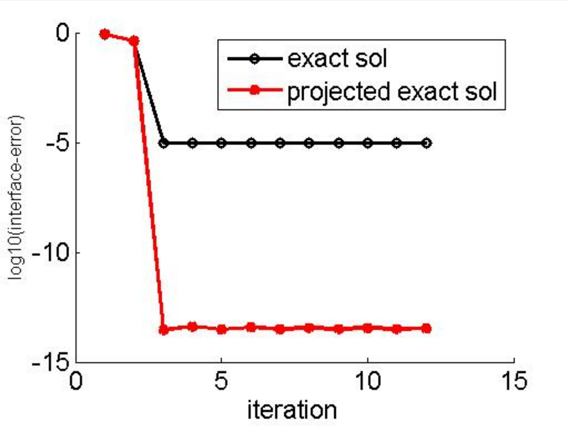
⇒ needs projection of interface solutions
⇒ use NUDFT for spectral interpolation :

$$u_2(\Gamma_2)(y_2) = \sum_{k=0}^{N_1} \hat{u}_{1,k}(\Gamma_2)\phi_{1,k}(y_2)$$





Aitken-Schwarz convergence for the Poisson problem on non uniform non conforming meshes for 100 (right) and 400 (left) interface points. The interface error is computed with respect to the exact discretised solution. $\Rightarrow$ Limitation not in the Aitken acceleration, rather on the solution representation

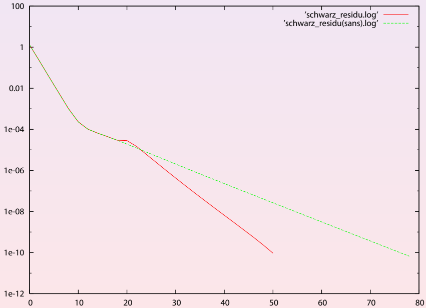


Aitken-Schwarz convergence for the Poisson problem on non uniform non conforming meshes for 100 interface points when the interface error is computed with respect to the exact discretised solution (black line) vs. the projected exact discretised solution (red line).

Outline

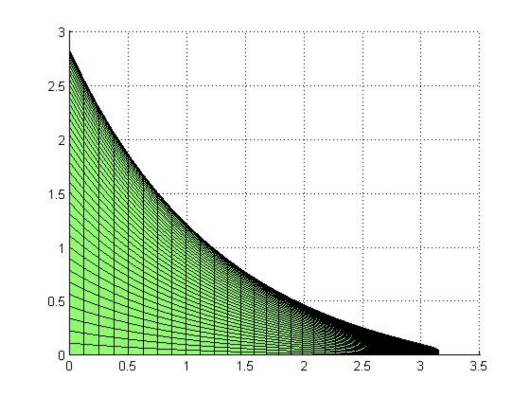
On going Work on the benchmark

- Re-write the communications in MPI standard of the C++ code liveDDM of V. Martin (2 months summer training period of P. Linel (2KE)).



On going Work on the benchmark

- Develop linear mapping between interfaces on regular data structure $\Rightarrow$ coupling of Fourier mode but adaptative building of the Aitken matrix is available (long term training project of P. Linel). Example linear convergence on nonuniform mesh $u_n = u_\infty + \sin(x - \pi) * \sin(y - \pi * (0.95)^i)$



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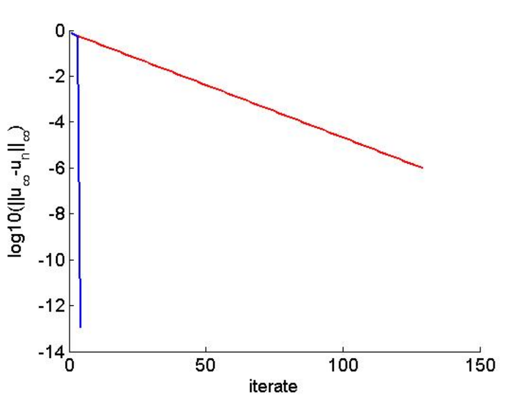

- implement the Aitken-Schwarz in the H2OLab software plateform developed at CAREN/IRISA for 3D flow in porous media with a lognormal distribution of the permeability with high heterogeneities.