A HOMOGENIZED MODEL OF AN UNDERGROUND NUCLEAR WASTE REPOSITORY INCLUDING THE POSSIBLY DAMAGED ZONE

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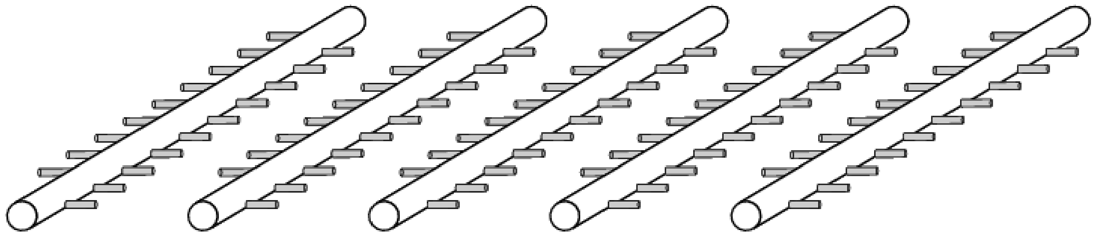

Figure 1: A part of a waste disposal unit, with 5 rows of containers and shafts

The Geometry

The whole domain: $\Omega =]0, L[^2 \times] - L/2, L/2[\subset \mathbf{R}^3$

Middle surface (on which the repositories are located) : $\Sigma =]0, L[^2 \times \{0\}$

Notations: $x = (x_1, x_2, x_3)$, $x' = (x_1, x_2)$, $y_i = x_i/\varepsilon$, $i = 2, 3$.

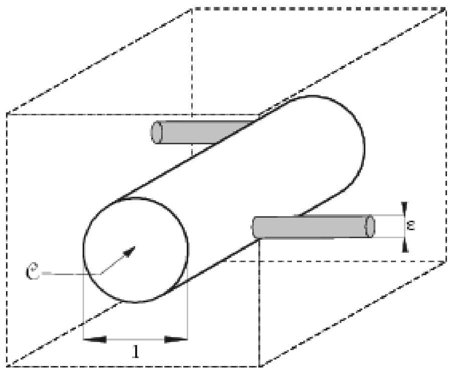

Figure 2: Cell of periodicity Y containing a shaft-damaged cylinder $\mathcal{S} =] - 1/2, 1/2[\times \mathcal{C}$ and a containers P

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The unit cell Y on figure **??** consists of three parts:

- The shaft-damaged cylinder $\mathcal{S} =]-1/2, 1/2[\times \mathcal{C}$ with basis $\mathcal{C}$
- The containers P
- The rest of the cell $\mathcal{Y}$

By the change of variable $y = x/\varepsilon$ we shrink the unit cell Y to the actual size ε and we denote by

$$Y_\varepsilon(i,j) = \varepsilon\ (\ (i,j,0) + Y)$$

the (i,j)-th ε-cell.
Inside these ε-cell:

- $\mathcal{P}_\varepsilon(i,j) = \varepsilon\ (\ (i,j,0) + P) =$ **the containers** set
- $\mathcal{S}_\varepsilon(i,j) = \varepsilon\ (\ (i,j,0) + \mathcal{S}) =$ **the shaft-damaged part**

Repeating Y_ε periodically over the surface Σ, we obtain the whole repository area and we denote :

$$\mathcal{P}_\varepsilon = \bigcup_{i,j=1}^{m} \varepsilon\ (\ (i,j,0) + P)\ \ \textbf{the union of all containers sets}$$

$$\mathcal{S}_\varepsilon = \bigcup_{i,j=1}^{m} \varepsilon\ (\ (i,j,0) + \mathcal{S})\ \ \textbf{the union of all shaft-damaged cylinders}$$

$$\mathcal{C}_\varepsilon = \bigcup_{i,j=1}^{m} \varepsilon\ (\ (i,j) + \mathcal{C})\ \ \textbf{the union of all shaft-damaged cylinders sections.}$$

The exterior boundary $S = \partial\Omega$ is divided in three parts:

$$\mathcal{F}_\varepsilon = \mathcal{S}_\varepsilon \cap \{x_1 = L\} \quad \textbf{the front of the shaft-damaged cylinders} \tag{1}$$
$$\mathcal{B}_\varepsilon = \mathcal{S}_\varepsilon \cap \{x_1 = 0\} \quad \textbf{the back of the shaft-damaged cylinders} \tag{2}$$
$$\mathcal{R}_\varepsilon = \mathcal{S}_\varepsilon \backslash (\mathcal{F}_\varepsilon \cup \mathcal{B}_\varepsilon) \quad \textbf{the rest of the exterior boundary of } \Omega\ . \tag{3}$$

The equations

$f^\varepsilon \in C([0,T] \times \Omega)$ - the function describing the sources (it has a compact support in time $[0,t_m] \subset]0,T[$ and in space $\mathcal{P}_\varepsilon$.)

$\lambda = \frac{\log 2}{\tau} > 0$, with τ being the half life of the radioactive element.

$\varphi_0 \in L^\infty(\Omega)$ - the initial concentration of the radioactive material in the soil.

The convection velocity $\mathbf{v}^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)$ **is larger in the shaft than in the host rock.** Thus we suppose that it has the form

$$\mathbf{v}^\varepsilon(x) = \begin{cases} \mathbf{v}^h(x) \ \text{ in the clay } \ \Omega\backslash\mathcal{S}_\varepsilon \\ \\ \varepsilon^{-\beta}\ \mathbf{v}^d(x', x_2/\varepsilon, x_3/\varepsilon) \ \text{ in the shaft } \ \mathcal{S}_\varepsilon \end{cases} .$$

We suppose for simplicity of this presentation that the convection velocity in shafts is unidirectional, i.e. that $\mathbf{v}^d = v^d(x_2, x_2/\varepsilon, x_3/\varepsilon)\,\mathbf{e}_1$.

The effective diffusion/dispersion tensor:

$$\mathbf{A}^\varepsilon = d\,\mathbf{I} + |\mathbf{v}^\varepsilon|\,\alpha^\varepsilon\,\mathbf{E}(\mathbf{v}^\varepsilon) \ ,$$

with

$$[\mathbf{E}(\mathbf{v}^\varepsilon)]_{ij} = \frac{v_i^\varepsilon\, v_j^\varepsilon}{|\mathbf{v}^\varepsilon|^2}$$

and

$$\alpha^\varepsilon = \begin{cases} \alpha^h & \text{in the host rock } \ \Omega\backslash\mathcal{S}_\varepsilon \\ \alpha^d & \text{in the shaft } \ \mathcal{S}_\varepsilon \end{cases} \quad , \quad .$$

We assume that

$$d(x) \geq d_0 > 0 \quad . \tag{4}$$

Thus

$$\mathbf{A}^\varepsilon(x) = \begin{cases} \mathbf{A}^h(x) \ \text{ in the host rock } \ \Omega\backslash\mathcal{S}_\varepsilon \\ \\ d(x)\,\mathbf{I} + \varepsilon^{-\beta}\,\mathbf{A}^d(x_2, x_2/\varepsilon, x_3/\varepsilon) \ \text{ in the shaft } \ \mathcal{S}_\varepsilon \end{cases} \quad .$$

Due to our assumption that the convection in shafts goes only in the direction of the shaft, **matrix $\mathbf{A}^d$ now has the form:**

$$\mathbf{A}^d(x_2, y_2, y_3) = |v^d|(x_2, y_2, y_3) \ \ (\,\mathbf{e}_1 \otimes \mathbf{e}_1\,)$$

Finally the porosity ω^ε is also larger in the shaft then in the host rock but the discrepancy is not so large as for the permeability

$$\omega^\varepsilon = \begin{cases} \omega^h & \text{in the host rock } \ \Omega\backslash\mathcal{S}_\varepsilon \\ \omega^d & \text{in the shaft } \ \mathcal{S}_\varepsilon \end{cases} \quad .$$

The problem:

$$\omega^\varepsilon\,\frac{\partial\varphi_\varepsilon}{\partial t} - \mathrm{div}\,(\mathbf{A}^\varepsilon\nabla\varphi_\varepsilon) + (\mathbf{v}^\varepsilon \cdot \nabla\,)\varphi_\varepsilon + \lambda\,\omega^\varepsilon\,\varphi_\varepsilon = f^\varepsilon \quad \text{in } \ \Omega^T \tag{5}$$

$$\varphi_\varepsilon(0,x) = \Phi_0(x) \ \ x \in \Omega \tag{6}$$

$$\tag{7}$$

We also need to impose some **boundary condition on the exterior boundary** $S = \partial\Omega$:

$$\mathbf{n}\cdot(\mathbf{A}^\varepsilon\nabla\varphi_\varepsilon - \mathbf{v}^\varepsilon\,\varphi_\varepsilon) = \kappa\ \ (\varphi_\varepsilon - g_\varepsilon) \quad \text{on } \ \mathcal{R}_\varepsilon^T \cup \mathcal{F}_\varepsilon^T \quad - \textbf{the Fourier law} \tag{8}$$

$$\varphi_\varepsilon = 0 \quad \text{on } \ \mathcal{B}_\varepsilon^T \quad - \textbf{back side is sealed} \tag{9}$$

$g_\varepsilon \in L^2(0,T;L^2(S))$ -the trace of the exterior concentration
$\kappa \in L^\infty(S)$, $0 < \kappa_0 \leq \kappa(x) \leq \kappa_1$ the rate of proportionality in the Fourier law.

Results:

Depending on β we distinguish 3 different cases :

$\beta < 1$ **This is the simplest case when the shafts do not make any contribution, i.e. the repository behaves as if they were not there.**

$\beta = 1$ **This is the most interesting case when the processes in the shaft and out of it are of the same order of intensity and there is an interaction between them.**

$\beta > 1$ **In this case the process in the shafts is dominant and we do not see the rest of the domain in the limit.**

A priori estimates

Proposition 1 *Let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem (??)-(??). Then there exists a constant $C > 0$, independent form ε, such that*

$$|\nabla\varphi_\varepsilon|_{L^2(0,T;L^2(\Omega))} \leq C \tag{10}$$

$$|\varphi_\varepsilon|_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{11}$$

$$|\varphi_\varepsilon|_{L^2(0,T;L^2(\mathcal{S}_\varepsilon))} \leq C\,\varepsilon^{\beta/2} \tag{12}$$

$$|\frac{\partial\varphi_\varepsilon}{\partial x_1}|_{L^2(0,T;L^2(\mathcal{S}_\varepsilon))} \leq C\,\varepsilon^{\beta/2} \quad . \tag{13}$$

The simplest case $\beta < 1$

The source: $f^\varepsilon(t,x) = \frac{1}{\varepsilon}\, f(t, \frac{x}{\varepsilon})$, $(y_1, y_2) \mapsto f(t, y_1, y_2, y_3)$ is 1- periodic *supp* $f \subset P \times [0, t_m]$

Theorem 1 *Let $\beta < 1$ and let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem (??)-(??). Then*

$$\varphi_\varepsilon \rightharpoonup \varphi \qquad \textit{weak* in} \quad L^\infty(0,T;L^2(\Omega)) \ \textit{ and weakly in} \quad L^2(0,T;H^1(\Omega)) \quad , \tag{14}$$

where φ is the unique solution of the problem

$$\omega^h\,\frac{\partial\varphi}{\partial t} - div\,(\mathbf{A}^h\nabla\varphi) + (\mathbf{v}^h\,\cdot\,\nabla\,)\varphi + \lambda\,\omega^h\,\varphi = \overline{f}\;\delta_\Sigma \quad in\ \ \tilde{\Omega}^T = (\Omega\backslash\Sigma)\times\,]0,T[\tag{15}$$

$$\varphi(x,0) = \Phi_0(x) \ \ x \in \tilde{\Omega} = \Omega\backslash\Sigma \ \ , \ \ \varphi = 0 \quad on \ \ S^T \tag{16}$$

$$\tag{17}$$

$$\overline{f} = \int_P f\ dy \quad .$$

The critical case $\beta = 1$

In this case the contributions from shafts and those from host rock are of the same order. Thus they both appear in the limit problem. We assume again that f^ε is of the same form $f^\varepsilon(t,x) = \frac{1}{\varepsilon}\, f(t, \frac{x}{\varepsilon})$.

$$g_\varepsilon = \begin{cases} g^h & \text{in the host rock } \mathcal{R}_\varepsilon \\ \varepsilon^{-1}\, g^d & \text{in the shaft } \mathcal{F}_\varepsilon \end{cases} \quad . \tag{18}$$

Two-scale convergence with respect to the singular measure:

In order to see some effects of the process that is going on in thin domain $\mathcal{S}_\varepsilon$ we need to consider the limit with respect to the rescaled measure $d\mu^\varepsilon(x) = \varepsilon^{-1}\, \mathbf{1}_{\mathcal{S}_\varepsilon}\, d\mathcal{L}_3$, where $d\mathcal{L}_3$ is the Lebesgue measure :

Definition 1 *A sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\varphi_\varepsilon \in L^p(\Omega)$ is said to converge two-scale to $\varphi_0 \in L^2(\Sigma \times \mathcal{C})$ if for any $\psi \in C(\Omega; L^{p'}(\mathcal{C})\,)$*

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi_\varepsilon(x)\psi(x, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})\, d\mu^\varepsilon(x) = \int_\Omega d\delta_\Sigma(x) \int_{\mathcal{C}} \varphi_0(x_1, x_2, y_2, y_3)\, \psi(x,y)\, dy_2\, dy_3 \ , \tag{19}$$

*In other words (**??**) can be simply written as*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathcal{S}_\varepsilon} \varphi_\varepsilon(x)\psi(x, \frac{x'}{\varepsilon})\, dx = \int_\Sigma dx_1\, dx_2 \int_{\mathcal{C}} \varphi_0(x_1, x_2, y)\, \psi(x_1, x_2, 0, y_2, y_3)\, dy \ .$$

We denote shortly

$$\varphi_\varepsilon \stackrel{2-d\mu}{\longrightarrow} \varphi_0$$

Theorem 2 *Let $\beta = 1$ and let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem (**??**)-(**??**). Then*

$$\varphi_\varepsilon \rightharpoonup \varphi \qquad \text{weak* in} \quad L^\infty(0,T; L^2(\Omega)\,) \ \text{ and weakly in } \ L^2(0,T; H^1(\Omega)) \ , \tag{20}$$

$$\varphi_\varepsilon \stackrel{2-d\mu}{\longrightarrow} \varphi(t, x_1, x_2, 0) \quad \text{two scale} \ , \tag{21}$$

where φ is the unique solution of the following variational problem :

Find $\varphi \in L^2(0,T;V)$, such that

$$\begin{aligned} &\int_0^T \int_\Omega \Big(-\omega^h\, \varphi\, \frac{\partial \psi}{\partial t} + \mathbf{A}^h \nabla\varphi\nabla\psi + (\mathbf{v}^h \cdot \nabla\,)\varphi\,\psi + \lambda\omega^h \varphi\,\psi \Big) + \int_{S^T} \kappa\varphi\psi + \\ &+ \int_0^T \int_\Sigma \Big(\langle |v^d| \rangle \frac{\partial \varphi}{\partial x_1}(t, x', 0)\, \frac{\partial \psi}{\partial x_1}(t, x', 0) + \langle v^d \rangle \frac{\partial \varphi}{\partial x_1}(t, x', 0)\, \psi(t, x', 0) \Big) = \\ &= \int_0^T \int_\Sigma \overline{f}\, \psi(t, x', 0) + \int_\Omega \omega^h\, \Phi_0\, \psi(0,x) + \int_{S^T} \kappa\psi\, g^h + \int_0^T \int_0^L \kappa\, \psi(t, L, x_2, 0)\, g^d \ , \\ &\textit{for any} \quad \psi \in H^1(0,T;V) \ \textit{ such that } \ \psi(T,x) = 0 \ . \end{aligned} \tag{22}$$

$$V = \{\phi \in H^1(\Omega)\ ;\ \frac{\partial}{\partial x_1}\phi(x_1,x_2,0) \in L^2(\Sigma)\ ,\ \phi(0,x_2,0) = 0\ \}\quad ,\quad \langle z \rangle = \int_{\mathcal{C}} z\, dy_2\, dy_3\quad .$$

Remark 1 *In case of non-diagonal* $\mathbf{A}^d$*, i.e. non-unidirectional convection in shafts, it is necessary to introduce* R*, the solution of the auxiliary problem*

$$\begin{aligned} &div_y\,(\mathbf{A}^d\,\nabla_y R) = -div_y\,(\mathbf{A}^d\,\mathbf{e}_1\,) \quad in\ \mathcal{C} \\ &\mathbf{n}\,\cdot\,\mathbf{A}^d\,\nabla_y\,R = -\mathbf{n}\,\cdot\,\mathbf{A}^d\,\mathbf{e}_1 \quad on\ \partial\mathcal{C}\ . \end{aligned} \tag{23}$$

to describe the oscillatory part of the two scale limit φ_1 *which has the form*

$$\varphi_1 = R(y)\,\frac{\partial \varphi_0}{\partial x_1}\quad .$$

Then the coefficient $\langle |v^d| \rangle$ *in the above problem mus be replaced by* $\langle \mathbf{A}^d \nabla_y (y_1 + R)\,\rangle$ *. In case of diagonal matrix* $\mathbf{A}^d$ *the solution of the auxiliary (**??**) is trivial* $R = 0$ *.*

The case of dominant shafts $\beta > 1$

In this last case the conductivity of shafts is so high that the process inside is dominant in the global scale. We assume that the behavior of g_ε is as follows:

$$g_\varepsilon = \begin{cases} g^h & \text{on the host rock } \mathcal{R}_\varepsilon \\ \varepsilon^{-\frac{\beta+1}{2}}\, g^d & \text{on the shaft } \mathcal{F}_\varepsilon \end{cases} \quad . \tag{24}$$

Remark 2 *We notice that in case* $\beta = 1$ *condition (**??**) reduces to the previously imposed condition on* g_ε*.*

Theorem 3 *Let* $\beta > 1$ *and let* $\{\varphi_\varepsilon\}$ *be the sequence of solutions to the problem (**??**)-(**??**). Then there exist a subsequence, denoted by the same symbol* $\{\varphi_\varepsilon\}$ *and an accumulation point* $\varphi^0 \in \mathcal{W} = \{\phi \in L^2(\Sigma \times \mathcal{C})\ ;\ \frac{\partial \phi}{\partial x_1} \in L^2(\Sigma \times \mathcal{C})\ \}$ *such that*

$$\varepsilon^{\frac{1-\beta}{2}}\,\varphi_\varepsilon \xrightarrow{2-d\mu} \varphi^0 \quad two\ scale \tag{25}$$

$$\varepsilon^{\frac{1-\beta}{2}}\,\frac{\partial \varphi_\varepsilon}{\partial x_1} \xrightarrow{2-d\mu} \frac{\partial \varphi^0}{\partial x_1} \quad two\ scale\ . \tag{26}$$

For $1 < \beta < 2$*, function* φ^0 *is the unique solution of the problem :*

$$\begin{aligned} &-\frac{\partial}{\partial x_1}\left(\langle |v^d| \rangle\,\frac{\partial \varphi^0}{\partial x_1}\right) + \langle v^d \rangle\,\frac{\partial \varphi^0}{\partial x_1} = 0 \quad in\quad]0,L[\\ &\varphi^0(0) = 0 \quad , \quad -\langle |v^d| \rangle\,\frac{\partial \varphi^0}{\partial x_1}(L) + (\langle v^d \rangle + \kappa)\,\varphi^0(L) = \kappa\,\langle g^d \rangle \quad . \end{aligned} \tag{27}$$

This problem can be solved explicitly and

$$\varphi^0 = \frac{\langle g^d \rangle}{1 + \frac{\langle v^d \rangle}{\kappa} - e^{sgn(v^d)\,L}}\,(1 - e^{sgn(v^d)\,x_1}) \tag{28}$$

For $\beta > 2$, function φ^0 is the unique solution of the problem :

$$-\frac{\partial}{\partial x_1}\left(|v^d|\,\frac{\partial \varphi^0}{\partial x_1}\right) + v^d\,\frac{\partial \varphi^0}{\partial x_1} = 0 \;\; in \;\;]0, L[\tag{29}$$

$$\varphi^0(0) = 0 \quad , \quad -|v^d|\,\frac{\partial \varphi^0}{\partial x_1}(L) + (v^d + \kappa)\,\varphi^0(L) = \kappa\, g^d \quad .$$

This problem can be solved explicitly too and

$$\varphi^0 = \frac{g^d}{1 + \frac{v^d}{\kappa} - e^{sgn(v^d)\,L}}\,(1 - e^{sgn(v^d)\,x_1}) \tag{30}$$

In case $\beta = 2$ and v^d independent on y, we have the convergence of mean concentrations in the shafts:

$$\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon} \varepsilon^{\frac{1-\beta}{2}}\,\varphi_\varepsilon \rightharpoonup \langle \varphi^0 \rangle \quad weakly\ in\ \ H^1(0, L) \ \ . \tag{31}$$

where

$$\varphi^0 = \frac{\langle g^d \rangle}{1 + \frac{v^d}{\kappa} - e^{sgn(v^d)\,L}}\,(1 - e^{sgn(v^d)\,x_1}) \tag{32}$$

Remark 3 *Actually, using the two-scale convergence, we can prove that, up to a subsequence $\varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0$ where φ^0 satisfies the partial differential equation*

$$-d\,\Delta_{y_2 y_3}\,\varphi^0 - |v^d|\,\frac{\partial^2 \varphi^0}{\partial x_1^2} + v^d\,\frac{\partial \varphi^0}{\partial x_1} = 0$$

in $]0, L[\, \times \mathcal{C}$ but we cannot identify the boundary conditions that it satisfies on $]0, L[\, \times \partial\mathcal{C}$. Thus, we cannot prove the uniqueness of the accumulation point.