Critical Scaling for Underground Waste Repository Modelling via Two-scale Convergence of Boundary Layers

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Scaling Up for Far Field Simulations of Underground Nuclear Waste in Performance Assessment

Couplex Model

We consider the following problem, considered for the couplex exercise

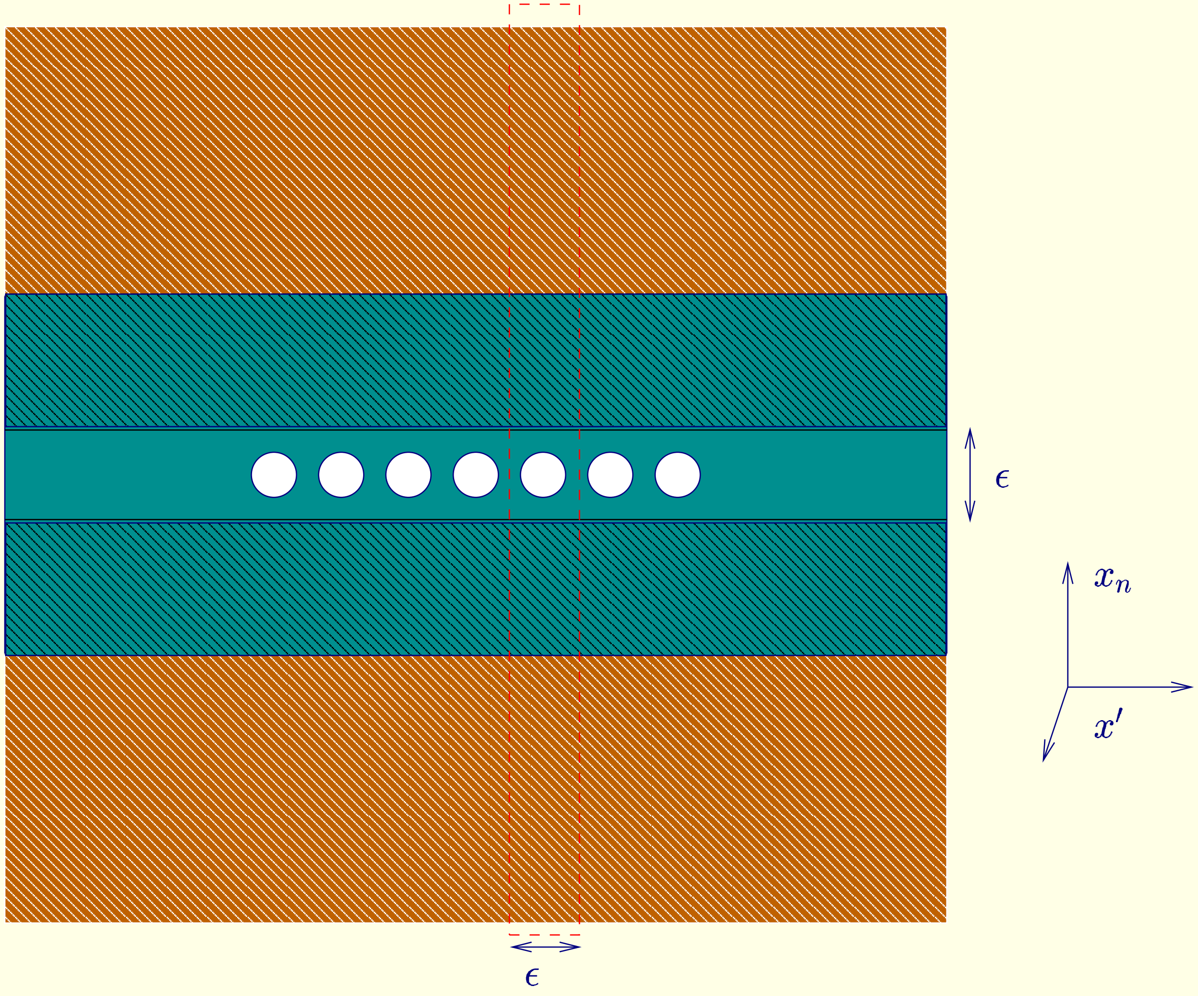
$$\begin{cases} \omega_\epsilon \frac{\partial \phi_\epsilon}{\partial t} - \mathrm{div}\,(A_\epsilon \nabla \phi_\epsilon) + \lambda \omega_\epsilon \phi_\epsilon = f_\epsilon(t,x) \text{ in } \Omega, \\ \phi_\epsilon = 0 \text{ on } \partial\Omega, \\ \phi_\epsilon(t=0) = \phi_0^\epsilon(x) \text{ in } \Omega. \end{cases}$$

where ϕ_ϵ represents a pollutant density.

What are the correct scales, i.e. the ones leading to an appropriate homogenized model

- taking into account short & long term behavior
- modelling sources concentrated in a small zone.

Geometry



Slow time variation and oscillatory profiles

The source term is of the form $f_\epsilon(t,x) = g\left(\epsilon^2 t, \frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right)$

$$A_\epsilon = \epsilon^2\left(a\left(\frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right) + B(x)\right)$$

where $\chi(y) = 1 \Leftrightarrow |y| < 1$.

Leads to a family of quasi-static problems in an infinite strip G

$$-\mathrm{div}\Big(\left(a(y)\chi(y_n) + B(0)\right)\nabla\psi(\tau,y)\Big) + \lambda\omega(y)\psi(\tau,y) = g(\tau,y)\chi(y_n)$$

with $\psi(y', y_n)$ periodic in y'.

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The factorized solution would be

$$\phi_\epsilon \approx u(t/\epsilon^2, x)\psi(t, \frac{x}{\epsilon}).$$

Non vanishing if $\lim_{y_n\to\pm\infty}\psi \not\longrightarrow 0.$, for example for $y \to \omega(y)$ with compact support.

In short,

$$\begin{cases} \omega_\epsilon \frac{\partial \phi_\epsilon}{\partial t} - \mathrm{div}\left(\left(a\left(\frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right) + B(x)\right)\nabla\phi_\epsilon\right) + \lambda\omega_\epsilon\phi_\epsilon = \frac{1}{\epsilon^2} f\left(\epsilon^2 t, \frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right) \text{ in } \Omega, \\ \\ \phi_\epsilon = 0 \text{ on } \partial\Omega, \\ \phi_\epsilon(t=0) = \phi_0^\epsilon(x) \text{ in } \Omega. \end{cases}$$

with $\omega_\epsilon(x) = \omega\left(\frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right)/\epsilon^2 + \omega_1(x)$. We then show that

$$\phi_\epsilon(t,x) = \phi_\epsilon(t/\epsilon^2, x)\psi(t, x/\epsilon), \text{ with } \phi_\epsilon \to u_0 \text{ solution of}$$

$$\begin{cases} -\mathrm{div}\left(B(x)\nabla u_0\right) + \lambda\omega_1(x)u_0 = 0 \text{ in } \Omega, \\ \\ u(x',0) = 1, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

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Quasi-static model

A related model (Sanchez-Palencia)

$$\begin{array}{rcl}\Delta u^{\epsilon} & = & f \text{ in } \Omega \cap \{|x_1| > \epsilon h\} \\ \epsilon a \Delta u^{\epsilon} & = & f \text{ in } \Omega \cap \{|x_1| < \epsilon h\} \\ u_{\epsilon} & = & 0 \text{ on } \partial\Omega.\end{array}$$

With transmission on the interface. Solved by matched asymptotics. In the strip,

$$u^{\epsilon} = v^0(y_1, x_2) + \epsilon v^1(y_1, x_2) + \ldots \text{ avec } y_1 = x_1/\epsilon$$

Which gives, at first order,

$$\frac{\partial^2 v^0}{\partial y_1^2} = 0 \rightsquigarrow v^0(y_1, x_2) = A(x_2) y_1 + B(x_2).$$

Transmission conditions yields

$$aA(x_2) = \frac{\partial u_1^0}{\partial x_1}(0, x_2) = \frac{\partial u_2^0}{\partial x_1}(0, x_2) \text{ and } u_1^0(0, x_2) - u_2^0(0, x_2) = A(x_2) 2h$$

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The resulting effective model is

$$\begin{aligned}
-\Delta u_1^0 &= f \text{ pour } x \in \Omega \cap \{x_1 < 0\} \\
-\Delta u_2^0 &= f \text{ pour } x \in \Omega \cap \{x_1 > 0\} \\
\text{et } \frac{\partial u_2^0}{\partial x_1}(0, x_2) = \frac{\partial u_2^0}{\partial x_1}(0, x_2) &= \frac{a}{2h}\left(u_1^0(0, x_2) - u(0, x_2)\right).
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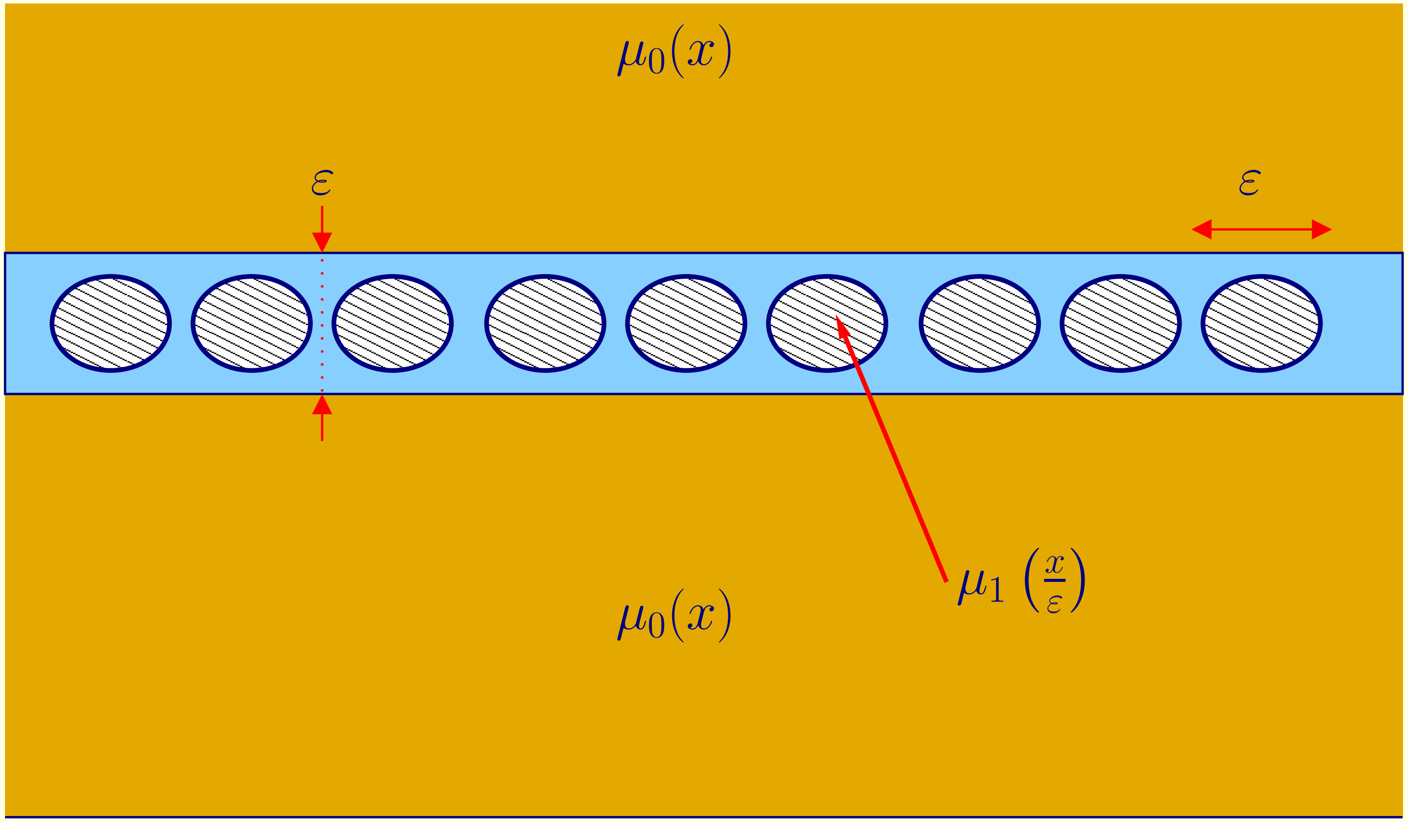
For a localized source, the corresponding limit model is $u = 0\ldots$

Alternative modelling

$$\begin{cases} \omega_\epsilon \frac{\partial \phi_\epsilon}{\partial t} - \operatorname{div}\left(\mu_\epsilon \nabla \phi_\epsilon\right) + \lambda \omega_\epsilon \phi_\epsilon = f_\epsilon\left(t, x\right) \chi\left(\frac{x_n}{\epsilon}\right) \text{ in } \Omega, \\ \phi_\epsilon = 0 \text{ on } \partial\Omega, \\ \phi_\epsilon(t=0) = \phi_0^\epsilon(x) \chi\left(\frac{x_n}{\epsilon}\right) \text{ in } \Omega. \end{cases}$$



A priori estimates

Assuming $\omega_\epsilon > c > 0$, we have

$$\|\phi_\epsilon\|_{L^\infty(0,T,L^2(\Omega))} + \left\|\sqrt{\mu_\epsilon}\nabla\phi_\epsilon\right\|_{L^2(0,T,L^2(\Omega)^N)} \leq C\sqrt{\epsilon}$$

Thus if $\mu_\epsilon > c > 0$ uniformly in ε, the homogenized equation is $u = 0$.

$\rightsquigarrow$ $\mu_\epsilon \to 0$ near the source.

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tool: *Two Scale convergence for boundary layers* (Allaire & Conca),

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega \phi_\epsilon(x)\psi\left(x', \frac{x}{\epsilon}\right) dx = \int_{x_n=0}\int_{Y'} \phi^1(x',y)\psi(x',y)dx'dy$$

for all $\psi \in \mathcal{D}\left(\mathbb{R}^{n-1}, C^\infty_\#([0,1]^{n-1}\times[0,+\infty))\right)$.

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If $\|\phi_\epsilon\|_{L^2(0,T,L^2(\Omega))} \leq C\sqrt{\epsilon}$ and $\|\nabla\phi_\epsilon\|_{L^2(0,T,L^2(\Omega))} \leq C\frac{1}{\sqrt{\epsilon}}$, then

$$\begin{aligned}\phi_\epsilon(t,x) &\overset{2SBL}{\rightharpoonup} \phi_0(t,x',y),\\ \epsilon\nabla\phi_\epsilon(t,x) &\overset{2SBL}{\rightharpoonup} \nabla_y\phi_0(t,x',y).\end{aligned}$$

Boundary Layer Scaling

Thus, to obtain a non trivial BL limit, we choose

$$\mu_\epsilon = \mu_0(x)\left(1-\chi\left(\frac{x_n}{\epsilon}\right)\right) + \epsilon^2\mu_1\left(x,\frac{x}{\epsilon}\right)\chi\left(\frac{x_n}{\epsilon}\right).$$

This leads to

$$\begin{array}{rcl} \phi_\epsilon(t,x) & \overset{2SBL}{\rightharpoonup} & \phi_1(t,x',y), \\ \epsilon\nabla\phi_\epsilon(t,x) & \overset{2SBL}{\rightharpoonup} & \nabla_y\phi_1(t,x',y). \end{array}$$

We show additionnally that

$$\left(1-\chi\left(\frac{x}{\epsilon}\right)\right)\phi_\epsilon(t,x) \rightharpoonup 0(2SBL),$$

and we obtain

$$\begin{array}{rcl} \omega(x',y)\dfrac{\partial\phi_1}{\partial t} - \mathrm{div}_y\left(\mu_1(x',y)\nabla\phi_1\right) + \lambda\omega(x',y)\phi_1 & = & f(t,x',y) \text{ for } |y_n|<1 \\ \phi_1(t,x',y',1) = \phi_1(t,x',y',-1) & = & 0 \\ y' \longrightarrow \phi_1(t,x',y',y_n) & & \text{periodic.} \end{array}$$

Macroscopic Behavior

The next order provides the macroscopic behavior:

Using the same method, $\frac{1}{\epsilon}\left(1-\chi\left(\frac{x}{\epsilon}\right)\right)u_\epsilon$ converges strongly in $L^2(0,T,L^2(\Omega))$ to u_1 and $\frac{1}{\epsilon}\left(1-\chi\left(\frac{x}{\epsilon}\right)\right)\nabla u_\epsilon$ converges weakly in $L^2(0,T,L^2(\Omega)^N)$ to ∇u_1, where u_1 is the unique solution of

$$\left\{\begin{aligned} \omega_0\left(x\right)\frac{\partial u_1}{\partial t} - \operatorname{div}\left(A_0(x)\nabla u_1\right) + \lambda\omega_0\left(x\right)u_1 + s^*(t)\delta_{x_n=0} &= 0 \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \partial\Omega, \\ u_1(t=0) &= 0 \quad \text{in } \Omega, \end{aligned}\right. \qquad (1)$$

where

$$s^*(t) = \int_{[0,1]^{n-1}}\left(\mu_1\left(y',1\right)\frac{\partial u_0}{\partial x_n}\left(t,y',1\right) - \mu_1\left(y',-1\right)\frac{\partial u_0}{\partial x_n}\left(t,y',-1\right)\right)dy'. \qquad (2)$$

An example

Resolution of a cell problem

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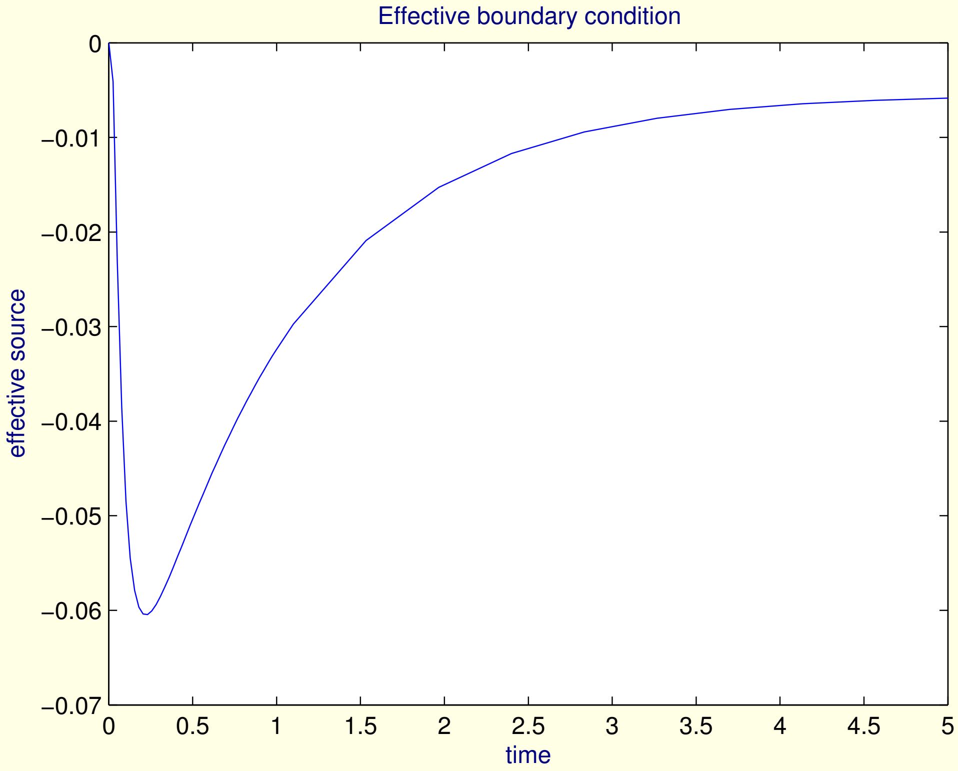

An example

The effective boundary condition is transmitted to the macroscopic equation (as a Neumann boundary condition). For a symmetric isotropic domain,

$$\begin{cases} \omega^* \dfrac{\partial u}{\partial t} - a^* \Delta u + \lambda \omega^* u = 0 \text{ in } \Omega \cap x_n > 0, \\[2em] u = 0 \text{ on } \partial\Omega \setminus \{x_n = 0\}, \\[2em] a^* \frac{\partial u}{\partial x_n} = \frac{1}{2} s^*(t) \text{ on } \{x_n = 0\}. \end{cases} \tag{3}$$