UN MODÈLE DE PROPAGATION D'ÉPIDEMIE

PAR CONTAMINATION DU SOL

LE SYSTÈME CHAT - PARVOVIRUS.

Michel LANGLAIS

UMR CNRS 5466, MATHÉMATIQUES APPLIQUÉES DE BORDEAUX,

UNIVERSITÉ VICTOR SEGALEN BORDEAUX 2,

et al.

D. Pontier, Université Lyon 1, UMR CNRS 5558

W.E. Fitzgibbon & J.J. Morgan, University of Houston
[twoside,12pt]article
amsfonts

1 The cat-FPLV homogeneous model

$$\begin{cases} \partial S/\partial t - d_s \Delta S = \\ b(S+I+R) - (m+kP)S - \sigma(S,I,R) - \rho CS - \nu S, \\ \partial I/\partial t - d_i \Delta I = \\ \qquad -(\alpha+\gamma)I - (m+kP)I + \sigma(S,I,R) + \rho CS, \\ \partial R/\partial t - d_r \Delta R = +e\alpha I - (m+kP)R + \nu S, \\ \partial C/\partial t = \phi(1-C)I - dC. \end{cases}$$

Initial conditions at $t = 0$:

$$S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0,\ R(x,0) = R_0(x) \geq 0,$$
$$0 \leq C(x,0) = C_0(x) \leq 1.$$

No flux boundary conditions on $\partial\Omega \times (0, +\infty)$:

$$d_s \nabla S \cdot \eta = d_i \nabla I \cdot \eta = d_r \nabla R \cdot \eta = 0.$$

Incidence term :

$$\sigma(S,I,R) = \begin{cases} \sigma_{pm} \dfrac{SI}{P}, & \text{proportionate mixing;} \\ \sigma_{ma} SI, & \text{mass action.} \end{cases}$$

Theorem 1 *Assume S_0, I_0 and R_0 lie in $(L^\infty(\Omega))^+$ and $0 \leq C_0(x) \leq 1$.*
The SIRC system has a unique, classical, global and component-wise nonnegative solution with $0 \leq C(x,t) \leq 1$.
Furthermore, when $k > 0$ there is a constant M such that

$$0 \leq S(x,t),\ I(x,t),\ R(x,t) \leq M < +\infty, \quad x \in \Omega,\ t > 0,$$

whereas for $k = 0$ there is a nondecreasing function $M : [0,+\infty) \to [0,+\infty)$ such that

$$0 \leq S(x,t),\ I(x,t),\ R(x,t) \leq M(T) < +\infty, \quad x \in \Omega,\ t \in [0,T].$$

Remark For $k = 0$ and $b - m - \alpha > 0$ trajectories are unbounded in $L^1(\Omega)$.

Proof

- getting $L^\infty(0,T; L^1(\Omega))$ estimates for $k = 0$ or $L^\infty(0,\infty; L^1(\Omega))$ estimates for $k > 0$;
- given a fix C with $0 \leq C(x,t) \leq 1$, deriving a priori $L^\infty((0,T) \times \Omega)$ estimates for $k = 0$ or $L^\infty((0,\infty) \times \Omega)$ estimates for $k > 0$; it uses duality arguments: M. Pierre, J.J. Morgan;
- conclusion, using the Leray Schauder fixed point Theorem, or semi-group theory, or else ...

2 A spatially heterogeneous domain

Ω in $I\!R^d,\ d \geq 1$, with smooth $\partial\Omega$.

population flux

$$-d(x)\nabla P + P\mathbf{c}(x),$$

$$d(x) = \begin{cases} d_i(x) & \text{for } x \in \Omega_i^*, \qquad\qquad d^*(\cdot) \in C(\overline{\Omega}_i^*),\ i = 1 \cdots K, \\ d_0(x) & \text{for } x \in \Omega - \bigcup_{l=1}^{K} \overline{\Omega}_i^*,\ d_0(\cdot) \in C(\Omega - \bigcup_{l=1}^{K} \overline{\Omega}_i^*). \end{cases}$$

(H1) continuous density P accross $\partial\Omega_i^*,\ i = 1 \cdots K$

(H2) balanced flux accross $\partial\Omega_i^*,\ i = 1 \cdots K$

$$[(d(x)\nabla P(x,t) - P(x,t)\mathbf{c}(x)) \cdot \eta_i(x)]_{\partial\Omega_i^*} = 0.$$

3 A heterogeneous S.I.R.C model

Densities $S(x,t) \geq 0$, $I(x,t) \geq 0$, $R(x,t) \geq 0$ and $0 \leq C(x,t) \leq 1$, with $P = S + I + R$.

- natural growth-rate: $b(x) - m(x) > r_{min} > 0$;
- density dependent growth-rate

  $$m(x) + k(x)P(x,t), \quad m(x) \geq 0,$$

  logistic effect when : $k(x) > k_{min} > 0$,
- incidence function : $\sigma(x, S, I, R)$, $\Omega = \Omega_{ma} \cup \Omega_{pm}$

  $$\begin{cases} \sigma_{ma}(x)SI, \ \sigma_{ma}(x) > 0, \ x \in \Omega_{ma}; \\ \sigma_{pm}(x)\dfrac{SI}{P}, \sigma_{pm}(x) > 0, \ x \in \Omega_{pm}. \end{cases}$$

- $\rho(x) \geq 0$ indirect transmission rate from environment to individuals,
  $\alpha(x) \geq 0$ additional death-rate of infective individuals,
  $\phi(x) \geq 0$, indirect transmission rate from individuals to environment,
  $d(x) \geq 0$, decontamination rate of environment;
- population fluxes $d_z(x) > d_{min} > 0, \ z = s, i, r$,

  $$-d_s(x)\nabla S, \ \ -d_i(x)\nabla I, \ \ -d_r(x)\nabla R;$$

Heterogeneous model on $\Omega \times (0, +\infty)$

$$\begin{cases} \partial S/\partial t - \mathrm{div}(d_s(x)\nabla S) = \\ \qquad b(x)P - (m(x) + k(x)P)S - \sigma(x, S, I, R) - \rho(x)CS, \\ \partial I/\partial t - \mathrm{div}(d_i(x)\nabla I) = \\ \qquad -\alpha(x)I - (m(x) + k(x)P)I + \sigma(x, S, I, R) + \rho(x)CS, \\ \partial R/\partial t - \mathrm{div}(d_r(x)\nabla R) = +e\alpha(x)I - (m(x) + k(x)P)R, \\ \partial C/\partial t = \phi(x)(1 - C)I - d(x)C. \end{cases}$$

Initial conditions at $t = 0$:

$$S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0,\ R(x,0) = R_0(x) \geq 0,$$
$$0 \leq C(x,0) = C_0(x) \leq 1.$$

No flux boundary conditions on $\partial\Omega \times (0, +\infty)$:

$$d_s(x)\nabla S \cdot \eta = d_i(x)\nabla I \cdot \eta = d_r(x)\nabla R \cdot \eta = 0.$$

Incidence term

$$\sigma(x, S, I, R) = \sigma_{ma}(x)SI\chi_{ma}(x) + \sigma_{pm}(x)\frac{SI}{P}\chi_{pm}(x).$$

Assumptions: bounded coefficients but $\mathbf{c}^{(j)} \in C^1(\overline{\Omega})^2$
logistic effect on mortality: $k(x) \geq k_{min} > 0$ on Ω.

Theorem 2 *There exists a unique nonnegative and classical solution* (S, I, R, C) *on* $\bar{\Omega} \times (0, \infty)$*, with* $0 \leq C(x,t) \leq 1$ *and*

$$0 \leq \|Z(\cdot, t)\|_{\infty,\Omega} \leq C_1(k_{min}) < +\infty,\ t > 0,\ Z = S, I, R;$$

Here $C_1(k_{min})$ *depends on* d_{min}*,* r_{min} *and the* $L^\infty(\Omega)$ *norm of the data.*

Proof

- getting $L^\infty(0, \infty; L^1(\Omega))$ estimates for $k_{min} > 0$;
- given a fix C with $0 \leq C(x,t) \leq 1$, deriving a priori $L^\infty((0,\infty) \times \Omega)$ estimates for $k_{min} > 0$;
  it uses ealier work by W.E. Fitzgibbon, M. Langlais and J.J. Morgan;
- conclusion, using the Leray Schauder fixed point Theorem, or semi-group theory, or else ...

4 Spatially $\varepsilon - \Theta$ periodic S.I.R.C. model

$$\begin{cases} \partial S/\partial t - \mathrm{div}(d_s(\frac{x}{\varepsilon})\nabla S) = \\ \qquad b(\frac{x}{\varepsilon})P - (m(\frac{x}{\varepsilon}) + k(\frac{x}{\varepsilon})P)S - \sigma(\frac{x}{\varepsilon}, S, I, R) - \rho CS, \\ \partial I/\partial t - \mathrm{div}(d_i(\frac{x}{\varepsilon})\nabla I) = \\ \qquad -\alpha(\frac{x}{\varepsilon})I - (m(\frac{x}{\varepsilon}) + k(\frac{x}{\varepsilon})P)I + \sigma(\frac{x}{\varepsilon}, S, I, R) + \rho(\frac{x}{\varepsilon})CS, \\ \partial R/\partial t - \mathrm{div}(d_r(\frac{x}{\varepsilon})\nabla R) = +e\alpha(\frac{x}{\varepsilon})I - (m(\frac{x}{\varepsilon}) + k(\frac{x}{\varepsilon})P)R, \\ \partial C/\partial t = \phi(\frac{x}{\varepsilon})(1 - C)I - d(\frac{x}{\varepsilon})C. \end{cases}$$

Initial conditions at $t = 0$:

$$S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0,\ R(x,0) = R_0(x) \geq 0,$$
$$0 \leq C(x,0) = C_0(x) \leq 1.$$

No flux boundary conditions on $\partial\Omega \times (0, +\infty)$:

$$d_s(\frac{x}{\varepsilon})\nabla S \cdot \eta = d_i(\frac{x}{\varepsilon})\nabla I \cdot \eta = d_r(\frac{x}{\varepsilon})\nabla R \cdot \eta = 0.$$

Behavior of $(S_\varepsilon, I_\varepsilon, R_\varepsilon, C_\varepsilon)$ **as** $\varepsilon \to 0$ **?**

Lemma 1 *As* $\varepsilon \to 0$, $(S_\varepsilon, I_\varepsilon, R_\varepsilon, C_\varepsilon) \to (S^\sharp, I^\sharp, R^\sharp, C^\sharp)$ *in* $L^2(\Omega \times (0,T))$, *where* $(S^\sharp, I^\sharp, R^\sharp, C^\sharp)$ *is the unique solution of a homogenized model*

$$\begin{cases} \partial S/\partial t - \operatorname{div}(\mathbf{D}_{hs}\nabla S) = b_0(S+I+R) - (m_0+k_0P)S \\ \qquad\qquad - \left[\mathcal{M}(\chi_{ma}\sigma_{ma})SI + \mathcal{M}(\chi_{pm}\sigma_{pm})\dfrac{SI}{P}\right] - \rho_0 CS, \\ \partial I/\partial t - \operatorname{div}(\mathbf{D}_{hi}\nabla I) = -\alpha_0 I - (m_0+k_0P)I \\ \qquad\qquad + \left[\mathcal{M}(\chi_{ma}\sigma_{ma})SI + \mathcal{M}(\chi_{pm}\sigma_{pm})\dfrac{SI}{P}\right] + \rho_0 CS, \\ \partial R/\partial t - \operatorname{div}(\mathbf{D}_{hr}\nabla R) = +e\alpha_0 I - (m_0+k_0P)R, \\ \partial C/\partial t = \phi_0(1-C)I - d_0 C. \end{cases}$$

satisfying the boundary conditions on $\partial\Omega \times (0, +\infty)$:

$$\mathbf{D}_{hs}\nabla S \cdot \eta = \mathbf{D}_{hi}\nabla I \cdot \eta = \mathbf{D}_{hr}\nabla R \cdot \eta = 0.$$

and the initial conditions at $t = 0$

$$S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0,\ R(x,0) = R_0(x) \geq 0,$$
$$0 \leq C(x,0) = C_0(x) \leq 1.$$

Notations $\varphi : I\!R^N \to I\!R$, Θ-periodic

$$\varphi_0 = \mathcal{M}(\varphi) = \frac{1}{|\Theta|}\int_\Theta \varphi(y)dy,$$
$$\delta^\sharp(p) = m_0 + k_0 p.$$